121p

[redacted]

NASA TT F-8224

INTERPOLATIVE-ANALYTICAL THEORY OF THE MOTION OF CERES

by M. S. Yarov-Yarovoy

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON May 1962

M. S. Yarov-Yarovoy

INTERPOLATIVE-ANALYTICAL THEORY OF THE MOTION OF CERES

Trudy Gosudarstvennovo Astronomicheskovo Instituta im. P.K. Shternberga 28, 25-90 (1960).

FOREWORD

The purpose of the present investigation was to apply the theory of interpolative-average variant of the three-point problem to the construction of the intermediate orbit of a specific celestial body, namely small planet No 1 (Ceres).

This is one of the components of a series of researches carried out by the Chair of Celestial Mechanics of the Moscow State University, devoted to the development of the analytical theory of secular and long-period perturbations on the basis of the method of averaging the perturbation function.

Since our work is the first attempt at a specific appplication of the theory of the interpolative-averaging variant of the three-point problem to a study of the motion of a small planet, we had to deal not only with the numerical processing of rather extensive observational material, and likewise with the sufficient development of

the theory of the intermediate perturbed motion of Ceres to permit comparison with observations, but also with a detailed development of the theory of the problem.

We call the reader's attention to the fact that the tables referred to heze are published on pp. 80 ff. of this volume.

It is our pleasant duty to express deep gratitude to the staff of the Chair of Celestial Mechanics and Gravimetry of the Moscow State University for the interest in our work.

CHAPTER I. NORMAL PLACES OF CERES AND THE SOLAR COORDINATES USED IN THE PRESENT WORK

1. Normal Places of Ceres Used in the Present Work

To construct an interpolative-analytic theory of the motion of Ceres, we used the observational material in the form of normal places, found in references [1, 2, 3, 4]. These normal places were reduced by us to the equator and to the equinox of 1950.0 and placed in Tables 1 and 2. We made use here of the tables of [5, 6].

In evaluating this observational material, we must say that it did not completely satisfy our requirement that is should have satisfied as a basis for the construction of a theory of the motion of Ceres. First, observa-

tion techniques have changed during the time interval from 1801-1938, i.e., almost a century and a half, and the accuracy in the determination of the positions of small planets has increased. Therefore the accuracy of the observational material is uneven. Second, during the last one hundred and forty years there have been certain changes in the techniques of data reduction, and such reduction constants as the precession constants, nutation constants, aberration constants, etc. have become known with great accuracy.

These two main factors, taken together, cannot help but reduce the usefulness of this observational material for the construction of an analytic theory for the motion of Ceres.

However, one cannot lose sight of the fact that during the last thirty or forty years the accuracy in observations of small planets has increased very slowly, and the reduction constants are now known sufficiently well for purposes of constructing an analytic theory for the motion of Ceres, and are fully reliable at the present-day observation accuracy. Therefore, it must be borne in mind in the construction of an analytic theory of the motion of Ceres that the results of the later observations (after 1938) while of course useful for comparison between theory and observation, can hardly change significantly the values

of the constants needed in the construction of an analytic theory for the motion of Ceres.

All this justifies our use of the normal places, obtained by the above-mentioned authors, for the construction of an analytic theory of the motion of Ceres. We confine ourselves in the present work to this observational material.

2. Solar Coordinates for the Epochs of the Normal Places

To process the observational material we have calculated the geocentric coordinates of the sun for the epochs of the normal places. In order not to overburden the text of the third chapter, we describe these calculations here.

We subdivided the work on the calculation of the solar coordinates into two parts, for the epochs of the normal place of the nineteenth and twentieth century respectively.

As regards the epochs of the normal places of the twentieth century, we have undertaken to calculate for them the solar coordinates on the basis of calculations directly from the Newcomb table [7]. The point is that in the almanacs of the twentieth century, particularly for the beginning of the century, the solar coordinates cannot be regarded as sufficiently reliable. We realized, of course, that this would be an exceedingly time consuming labor, but there was no way out. Quite unexpectedly, how-

ever, this work was made even more complicated by the fact that Newcomb's tables for the motion of the sun contain many misprints, the detection and elimination of which called for additional labor. We shall discuss these misprints in Newcomb's tables in the next section.

As regards the solar coordinates for the epochs of the normal places of Ceres in the twentieth century, we have copied them from the corresponding volumes of the "Nautical Almanac." They are subject to no doubt.

All the solar coordinates were then reduced to the ecliptic and to the 1950.0 equinox. The results of these calculations are given by us in Table 6.

3. Misprints in Newcomb's Tables of Solar Motion

During the computations we observed many misprints in Newcomb's tables of solar motion [7]. We give here a list of these misprints.

1. Pages 36 and 37, 1908. The period has not been eliminated from the argument D, while the quantities VI and VII have been calculated under the assumption that this period has been eliminated. The value of D should therefore be 0.181 in place of the 29.712 printed in the tables.

2. On the same pages, the quantity 1 for years 1940B, 1942, and 1943 is incorrectly given (see the second column of the following small table). The correct values are

listed in the third column of the following table:

Years	Erroneously given in the tables	Correct values
1940B	19.3	9.3
1942	26.6	16.6
1943	0.2	20.2

3. Pages 37 and 38. The argument M for 1950 on page 37 is 2.8406, while on page 38 it is 2.8405. The correct values of M are 2.8406.

4. On page 41 for September and October the dates are tangled up, although the day of the year, and also the values of L and τ, are correct. Therefore in place of the printed dates 1--4, 19--30, and 5--30 for September and 1--18, and 31 for October we should have 1--30 for September and 1--31 for October. As was pointed out to us by V. F. Proskurin, this misprint is not contained in all the copies of the Newcomb table.

5. On page 67: Vertical argument 24, horizontal argument 116 -- should be 2395 (printed 2495).

6. On page 73: Vertical argument 16, horizontal argument 5 -- should be 75 (printed 85).

7. Page 125: Vertical argument 368, horizontal argument 68 -- should be 1875 (printed 1975).

4. Comparison of the Solar Coordinates Calculated by Us and Those Calculated by Other Authors

Our calculated solar coordinates can be compared in part with the solar coordinates found in the article by V. F. Proskurin and T. I. M[illegible]shinskaya [4] and in the article by N. V. Kamendantov [8]. The values of the coordinates coincide in all cases for the 20-th century, and in almost all cases for the 19-th century (except 1836 and 1880). As regards 1836 and 1880, which are the only leap years of the 19-th century for which the solar coordinates have been calculated in [4] and [8], the discrepancy in the solar coordinates for these epochs lies in the sixth decimal place. A recalculation disclosed no error whatever in our calculations. We are therefore inclined to assume that some features involved in the use of the Newcomb tables for leap years have not been taken into consideration in [4] and [8].

CHAPTER II. TRIPLY AVERAGED ORTHOINTERPOLATIONAL VARIANT OF SPATIAL ECLIPTIC LIMITED THREE-POINT PROBLEM

1. Differential Equations of Perturbed Motion of Triply Averaged Interpolative Variant of Spatial Ecliptic Limited Three-Point Problem and Their Integrals

Orthointerpolational Condition

We shall consider the problem of the perturbed motion of Ceres as a limited Newtonian three-dimensional elliptical problem of three points: -- sun -- Jupiter -- asteroid (ceres).

The state of motion of the material point P, representing Ceres, will be assumed defined by an aggregate of canonical elements

$$x_1, \ x_2, \ x_3; \ y_1, \ y_2, \ y_3. \qquad (2.1)$$

Let the characteristic function of the problem be

$$\Omega = \Omega(x_1, \ x_2, \ x_3; \ y_1, \ y_2, \ y_3; \ t). \qquad (2.2)$$

Then the perturbed motion of the small planet P will be determined by a canonical system of six differential equations of the following form:

$$\frac{dx_i}{dt} = \frac{\partial\Omega}{\partial y_i}, \ \frac{dy_i}{dt} = \frac{\partial\Omega}{\partial x_i}, \ i = 1, 2, 3. \qquad (2.3)$$

Let the coordinates x_1, x_2, x_3 and the momenta y_1, y_2, and y_3 be so chosen that by processing the observational material, which us provides us with discrete values of these variables for a series of successive instants of time t from a certain range of variation of the time $(\underline{t}, \overline{t})$, we have obtained more probable constant values of the

quantities l_1, l_2, l_3; $m_1^{(1)}$, $m_3^{(1)}$, $m_2^{(2)}$, $m_3^{(2)}$, and $\tilde{\lambda}$, $\tilde{\mu}_1$, and $\tilde{\mu}_2$ in the expressions:

$$l_1 x_1 + l_2 x_2 + l_3 x_3 = \tilde{\lambda}, \tag{2.4}$$

$$m_1^{(1)} y_1 + m_3^{(1)} y_3 = \tilde{\mu}_1, \tag{2.5}$$

$$m_2^{(2)} y_2 + m_3^{(2)} y_3 = \tilde{\mu}_2. \tag{2.6}$$

Considering now the coefficients l_1, l_2, l_3; $m_1^{(1)}$, $m_3^{(1)}$, $m_2^{(2)}$, and $m_3^{(2)}$ in relations (2.4), (2.5), and (2.6) as being constant, and λ, μ_1, and μ_2 as functions of the variables (2.1), determined from the formulas

$$l_1 x_1 + l_2 x_2 + l_3 x_3 = \lambda, \tag{2.7}$$

$$m_1^{(1)} y_1 + m_3^{(1)} y_3 = \mu_2, \tag{2.8}$$

$$m_2^{(2)} y_2 + m_3^{(2)} y_3 = \mu_2, \tag{2.9}$$

we can eliminate some of the variables contained in the characteristic function Ω, expressing them with the aid of (2.7), (2.8), and (2.9) in terms of λ, μ_1, μ_2, and other variables.

We shall call the quantities λ, μ_1, and μ_2 the interpolation elements, while $\tilde{\lambda}$, $\tilde{\mu}_1$, and $\tilde{\mu}_2$ will be called their most probable values within the limits of the empirical material under consideration.

We assume that we have eliminated the variables x_1, y_1, and y_2. In this case we have

$$\Omega = \Omega(x_1, x_2\, x_3;\ y_1, y_2, y_3;\ t) =$$
$$= \Omega[x_1(\lambda, x_2, x_3), x_2, x_3;\ y_1(\mu_1, y_3), y_2(\mu_2, y_3), y_3;\ t] = \tag{2.10}$$
$$= \Omega^*(\lambda, \mu_1, \mu_2;\ x_2, x_3;\ y_3;\ t),$$

where Ω^* denotes Ω as a function of the variables λ, μ_1, μ_2; x_2, x_3; y_3; t.

Let the coordinate x_2 in the time interval $(\underline{t}, \overline{t})$ vary from $\underline{x}_2$ (lower limit) to $\overline{x}_2$ (upper limit). We shall denote the similar lower and upper limits of the variation of the quanties x_3 and y_3 in the time interval $(\underline{t}, \overline{t})$ respectively by $\underline{x}_3$, $\overline{x}_3$ and $\underline{y}_3$, $\overline{y}_3$.

Let us average our function Ω^* over the variables x_2, x_3, and y_3:

$$\widetilde{\Omega} = \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int\limits_{x_2=\underline{x}_2}^{\overline{x}_2} \int\limits_{x_3=\underline{x}_3}^{\overline{x}^3} \int\limits_{y_3=\underline{y}_3}^{\overline{y}^3} \Omega^*\, dx_2\, dx_3\, dy_3, \tag{2.11}$$

where in the process of integration of the function Ω^* we assume that λ, μ_1, and μ_2 are independent of x_2, x_3, and y_3. The averaged characteristic function, which we denote by $\widetilde{\Omega}$, will depend only on λ, μ_1, μ_2, and t:

$$\widetilde{\Omega} = \widetilde{\Omega}(\lambda, \mu_1, \mu_2;\ t). \tag{2.12}$$

After averaging we shall assume that the function Ω will depend also on the variables x_1, x_2, x_3; y_1, y_2, y_3 but only via the interpolation elements λ, μ_1, and μ_2,

in accordance with (2.7), (2.8), and (2.9).

We now substitute the function $\widetilde{\Omega}$ in place of the function Ω in equations (2.3). This gives us the equations for the triply-averaged interpolated variant of our problem

$$\frac{dx_i}{dt} = \frac{\partial\widetilde{\Omega}}{\partial y_i}, \frac{dy_i}{dt} = -\frac{\partial\widetilde{\Omega}}{\partial x_i}. \tag{2.13}$$

As already noted, the function $\widetilde{\Omega}$ depends on the variables x_1, x_2, x_3; y_1, y_2, y_3 only through the quantities λ, μ_1, and μ_2. We can therefore rewrite (2.13) in the form (taking (2.7), (2.8), and (2.9) into account)

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= \frac{\partial\widetilde{\Omega}}{\partial\mu_1}\cdot m_1^{(1)}, & \frac{dy_1}{dt} &= -l_1\frac{\partial\widetilde{\Omega}}{\partial\lambda},\\
\frac{dx_2}{dt} &= \frac{\partial\widetilde{\Omega}}{\partial\mu_2}\cdot m_2^{(2)}, & \frac{dy_2}{dt} &= -l_2\cdot\frac{\partial\widetilde{\Omega}}{\partial\lambda},\\
\frac{dx_3}{dt} &= \frac{\partial\widetilde{\Omega}}{\partial\mu_1}\cdot m_3^{(1)} + \frac{\partial\widetilde{\Omega}}{\partial\mu_2}\cdot m_3^{(2)}, & \frac{dy_3}{dt} &= -l_3\cdot\frac{\partial\widetilde{\Omega}}{\partial\lambda}.
\end{aligned}\right\} \tag{2.14}$$

These equations have three first integrals. They can be determined in the following manner.

We multiply the first equation of the system (2.14) by $m_3^{(1)}m_2^{(2)}$, the second by $m_1^{(1)}m_3^{(2)}$, and the third by $-m_1^{(1)}m_2^{(2)}$. We obtain

$$m_3^{(1)}m_2^{(2)}\frac{dx_1}{dt} + m_1^{(1)}m_3^{(2)}\frac{dx_2}{dt} - m_1^{(1)}m_2^{(2)}\frac{dx_3}{dt} = \left(m_1^{(1)}m_3^{(1)}m_2^{(2)} - m_3^{(1)}m_1^{(1)}m_2^{(2)}\right)\frac{\partial\widetilde{\Omega}}{\partial\mu_1} +$$

$$+\left(m_2^{(2)}m_1^{(1)}m_3^{(2)} - m_3^{(2)}m_1^{(1)}m_2^{(2)}\right)\frac{\partial\widetilde{\Omega}}{\partial\mu_2} = 0. \qquad (2.15)$$

Integrating (2.15) we obtain

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ m_1^{(1)} & 0 & m_3^{(1)} \\ 0 & m_2^{(2)} & m_3^{(2)} \end{vmatrix} = \mathfrak{A}_1. \qquad (2.16)$$

Multiplying now the fourth equation of (2.14) by l_3 and the sixth by $-l_1$, and adding, we get

$$l_3\frac{dy_1}{dt} - l_1\frac{dy_3}{dt} = (-l_1 l_3 + l_3 l_1)\frac{\partial\widetilde{\Omega}}{\partial\lambda} = 0. \qquad (2.17)$$

From this we obtain

$$l_1 y_3 - l_3 y_1 = \mathfrak{A}_2. \qquad (2.18)$$

We can similarly obtain

$$l_2 y_3 - l_3 y_2 = \mathfrak{A}_3. \qquad (2.19)$$

31 Equations (2.16), (2.18), and (2.19) determine our first three integrals of the system (2.14).

The existence of the three first integrals (2.16), (2.18), and (2.19) is the consequence of the fact that we have replaced the function Ω by its triply-averaged value $\widetilde{\Omega}$. It is easy to see that were we to carry out similarly only single or double averaging of the charac-

teristic function Ω, we would obtain only one or two first integrals respectively.

A particularly interesting case is when the conditions

$$m_1^{(1)} l_1 + m_3^{(1)} l_3 = 0, \quad m_2^{(2)} l_2 + m_3^{(2)} l_3 = 0, \tag{2.20}$$

are satisfied; these are called the orthointerpolational conditions.

Indeed, in this case we have

$$\left.\begin{aligned} \frac{d\lambda}{dt} = l_1 \frac{dx_1}{dt} + l_2 \frac{dx_2}{dt} + l_3 \frac{dx_3}{dt} = \left(l_1 m_1^{(1)} + l_3 m_3^{(1)}\right) \frac{\partial \widetilde{\Omega}}{\partial \mu_1} + \\ + \left(l_2 m_2^{(2)} + l_3 m_3^{(2)}\right) \frac{\partial \widetilde{\Omega}}{\partial \mu_2} = 0, \\ \frac{d\mu_1}{dt} = -\left(l_1 m_1^{(1)} + l_3 m_3^{(1)}\right) \frac{\partial \widetilde{\Omega}}{\partial \lambda} = 0, \quad \frac{d\mu_2}{dt} = -\left(l_2 m_2^{(2)} + l_3 m_3^{(2)}\right) \frac{\partial \widetilde{\Omega}}{\partial \lambda} = 0, \end{aligned}\right\} \tag{2.21}$$

i.e., the system (2.14) has integrals

$$\lambda = \text{const}, \quad \mu_1 = \text{const}, \quad \mu_2 = \text{const}. \tag{2.22}$$

In the case when the orthointerpolational conditions (2.20) are satisfied, the integrals (2.22) coincide with (2.16), (2.18), and (2.19). In fact, the integral (2.16) can be written in the form

$$\frac{m_3^{(1)}}{m_1^{(1)}} \cdot x_1 + \frac{m_3^{(2)}}{m_2^{(2)}} \cdot x_2 - x_3 = \frac{\mathfrak{A}_1}{m_1^{(1)} m_2^{(2)}}. \tag{2.23}$$

Using the orthointerpolational conditions (2.20) we can rewrite this equation in the form

$$-\frac{l_1}{l_3}\cdot x_1 - \frac{l_2}{l_3}\cdot x_2 - x_3 = \frac{\mathfrak{A}_1}{m_1^{(1)} m_2^{(2)}}, \tag{2.24}$$

or

$$l_1 x_1 + l_2 x_2 + l_3 x_3 = -\frac{l_3 \mathfrak{A}_1}{m_1^{(1)} m_2^{(2)}} = \lambda. \tag{2.25}$$

Similarly, from the integral λ = const we can obtain the integral (2.16) by using (2.25), (2.24), (2.20), and (2.23).

Let us show now that the integral μ_1 = const is the consequence of the integral (2.18) and equation (2.20). Indeed, equation (2.18) can be rewritten in the form

$$\frac{l_1}{l_3}\cdot y_3 - y_1 = \frac{\mathfrak{A}_2}{l_3}. \tag{2.26}$$

$$-\frac{m_3^{(1)}}{m_1^{(1)}}\cdot y_3 - y_1 = \frac{\mathfrak{A}_2}{l_3}, \tag{2.27}$$

or

$$m_1^{(1)} y_1 + m_3^{(1)} y_3 = -\frac{\mathfrak{A}_2}{l_3}\cdot m_1^{(1)} = \mu_1. \tag{2.28}$$

Similarly we obtain (2.18) from the integral μ_1 = const, using (2.28), (2.27), (2.20), and (2.26).

Analogously, the integral μ_2 = const is obtained from (2.19) and vice-versa.

The orthointerpolational case is quite remarkable.

Indeed, relations (2.7), (2.8), and (2.9), which we have called the empirical integrals, are in the orthointerpolational case not only empirical, but actual integrals of the system (2.13) or of its equivalent (2.14), obtained after interpolative averaging of the characteristic function. In the same case, if the orthointerpolational conditions are not satisfied, the empirical integrals (2.7), (2.8), and (2.9) are not the first integrals of the system (2.14). In fact, in this case, as we have already seen, there exist the first integrals

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ m_1^{(1)} & 0 & m_3^{(1)} \\ 0 & m_2^{(2)} & m_3^{(2)} \end{vmatrix} = \mathfrak{A}_1, \tag{2.16}$$

$$l_1 y_3 - l_3 y_1 = \mathfrak{A}_2 \tag{2.18}$$

$$l_2 y_3 - l_3 y_2 = \mathfrak{A}_3. \tag{2.19}$$

But we have taken as the basis of the interpolative averaging

$$l_1 x_1 + l_2 x_2 + l_3 x_3 = \lambda, \tag{2.7}$$

$$m_1^{(1)} y_1 \qquad + m_3^{(1)} y_3 = \mu_1 \tag{2.8}$$

$$m_2^{(2)} y_2 + m_3^{(2)} y_3 = \mu_2. \tag{2.9}$$

Comparing (2.16), (2.18), and (2.19) with (2.7), (2.8), and (2.9) we find that when these equations are compatible we get

$$\left.\begin{array}{l} m_1^{(1)} l_1 + m_3^{(1)} l_3 = 0, \\ m_2^{(2)} l_2 + m_3^{(2)} l_3 = 0. \end{array}\right\} \qquad (2.29)$$

However, since these equations coincide with the orthointerpolational conditions (2.20), which are not satisfied in this case, our empirical integrals (2.7), (2.8), and (2.9) are not the first integrals of the system (2.14), and even contradict them. Therefore, in the orthointerpolational case the use of relations (2.7), (2.8), and (2.9) and the subsequent determination of the characteristic function is incomparably more valid than in the case when the orthointerpolational conditions are not satisfied. In this lies the closure property of the canonical system (2.3) in the orthointerpolational case, for here the relations (2.7), (2.8), and (2.9), obtained by processing empirical material that yields solutions of the system (2.3) in general with a certain approximation, are also first integrals of the system (2.14).

The presence of the integrals (2.22) enables us to readily complete the integration of the system (2.14) in quadratures.

Namely, we have

16

$$
\begin{aligned}
x_1 &= x_{1n} + m_1^{(1)} \cdot \frac{\partial \underline{\Omega}}{\partial \mu_1}, & y_1 &= y_{1n} - l_1 \cdot \frac{\partial \underline{\Omega}}{\partial \lambda}, \\
x_2 &= x_{2n} + m_2^{(2)} \cdot \frac{\partial \underline{\Omega}}{\partial \mu_2}, & y_2 &= y_{2n} - l_2 \cdot \frac{\partial \underline{\Omega}}{\partial \lambda}, \\
x_3 &= x_{3n} + m_3^{(1)} \cdot \frac{\partial \underline{\Omega}}{\partial \mu_1} + m_3^{(2)} \cdot \frac{\partial \underline{\Omega}}{\partial \mu_2}, & y_3 &= y_{3n} - l_3 \cdot \frac{\partial \underline{\Omega}}{\partial \lambda},
\end{aligned}
\tag{2.30}
$$

where

$$
\underline{\Omega} = \int\limits_{t=t_0}^{t} \widetilde{\Omega}(\lambda, \mu_1, \mu_2; t)\, dt, \tag{2.31}
$$

and λ, μ_1, μ_2 are independent of t, while x_{1n}, x_{2n}, x_{3n}; y_{1n}, y_{2n}, y_{3n} are the initial values of the corresponding variables when $t = t_0$.

2. Employed System of Canonical Elements

We shall analyze the motion of Ceres in canonical variables, which are close in their structure to the first system of Poincare's canonical variables. They can be expressed in the following fashion in terms of the ordinary Kepler osculating elements

$$
\begin{aligned}
x_1 &= k\sqrt{a}, & y_1 &= M + \omega + \Omega - l_j, \\
x_2 &= k(\sqrt{a} - \sqrt{p}), & y_2 &= l_j - \omega - \Omega, \\
x_3 &= k(\sqrt{p} - \sqrt{p} \cdot \cos\gamma), & y_3 &= l_j - \Omega.
\end{aligned}
\tag{2.32}
$$

Here: k -- Gaussian constant (we set the mass of the sun equal to unity);

a -- major semi-axis,

p -- parameter,

γ -- inclination of the plane of the osculating orbit of Ceres to the plane of the orbit of Jupiter,

M -- average anomaly,

ω -- angular distance of the perihelium from the node,

Ω -- longitude of the ascending node of the osculating orbit of Ceres,

l_j -- average longitude of Jupiter in the orbit.

By node of the osculating orbit of Ceres we mean here the ascending node of the orbit of Ceres not relative to the ecliptic, but relative to the plane of the orbit of Jupiter.

We assume Jupiter to move in its unperturbed orbit with the elements as calculated by Hill [14]:

Ecliptic and equinox, 1850, January, 0.0

$$\left.\begin{aligned} M_{0j} &= 148°01'58''{,}33 \\ \Omega_j &= 98\ 55\ 58\ {,}16 \\ \omega_j &= 272\ 58\ 28\ {,}56 \\ i_j &= 1\ 18\ 41\ {,}81 \\ e_j &= 0{,}04825382 \\ a_j &= 5{,}2028029 \\ m_j &= 1{:}1047{,}355 \end{aligned}\right\} \quad (2{,}33)$$

The average longitude of Jupiter l_j and the longitude of the node of Ceres Ω are reckoned from one and the same direction in the plane of the Jupiter orbit.

The system of canonical variables which we have introduced is suitable only for the limited three-point problem. In this case it has certain advantages over the corresponding first system of Poincare's canonical vari-

ables. These advantages will become particularly obvious when the perturbation function is expanded. On the other hand, the difference between the variables (2.32) which we have introduced and Poincare's first system lies in the fact that all three generalized momenta contain the terms f_j. This leads, of course, also to a change in the form of the characteristic function of the problem.

Obviously, if we find the time dependence of the variables x_i and y_i ($i = 1, 2, 3$), then we obtain also the dependence of the rectangular heliocentric coordinates and velocity components on the time, by virtue of the fact that, apart from special exceptional cases, a one-to-one correspondence exists between the variables x_i and y_i ($i = 1, 2, 3$) and the rectangular heliocentric coordinates and components of the velocity, via the osculating elements.

3. Form of the Characteristic Function Ω in the Limited Three-Dimensional Elliptical Problem of Three Points

The Characteristic function Ω for the first system of Poincare's canonical variables has in the limited three-point problem the form [15]:

$$k^2 m_j \left(\frac{1}{\Delta} - \frac{r \cos \vartheta}{r_j^2} \right) + \frac{k^2}{2a} . \qquad (2.34)$$

19

Here:

Δ -- mutual distance between the small planet and Jupiter,

r_j -- heliocentric distance of Jupiter,

r -- heliocentric distance of the small planet,

ϑ -- angle between the direction from the sun to the small planet and to Jupiter,

m_j -- mass of Jupiter, expressed in units of solar mass.

The quantities Δ, r_j, r, and ϑ are assumed expressed in terms of the Poincare canonical variables:

$$\left.\begin{array}{ll} L = k\sqrt{a}, & \lambda = M + \omega + \Omega, \\ \rho_1 = k(\sqrt{a} - \sqrt{p}), & \omega_1 = -\omega - \Omega, \\ \rho_2 = k(\sqrt{p} - \sqrt{p}\cos\gamma), & \omega_2 = -\Omega, \end{array}\right\} \tag{2.35}$$

the constant parameters of the Jupiter orbit and the time t.

The canonical variables introduced by us (2.32) are expressed in terms of Poincare's elements in the following fashion

$$\left.\begin{array}{ll} x_1 = L, & y_1 = \lambda - l_j, \\ x_2 = \rho_1, & y_2 = \omega_1 + l_j, \\ x_3 = \rho_2, & y_3 = \omega_2 + l_j. \end{array}\right\}. \tag{2.36}$$

By virtue of the fact that

$$\frac{dl_j}{dt} = n_j, \qquad (2.37)$$

where n_j is a constant, the characteristic function for the variables (2.32) in the limited three-point problem has the form

$$\Omega = \frac{k^4}{2x_1^2} + n_j(x_1 - x_2 - x_3) + k^2 m_j\left(\frac{1}{\Delta} - \frac{r\cos\vartheta}{r_j^2}\right), \qquad (2.38)$$

where the third term in the first part, usually called the perturbation function, is expressed in terms of the variables (2.32), the constant parameters of the Jupiter orbit, and the time t.

4. Explicit Expression for the Perturbation Function in Terms of the Canonical Variables (2.32)

We denote the third term in the right half of (2.38) by $k^2 m_j W_j$. Here, obviously,

$$W_j = \frac{1}{\Delta} - \frac{r\cos\vartheta}{r_j^2}. \qquad (2.39)$$

Le Verrier [16] obtained the following explicit expression for the function W_j in terms of the Kepler osculating elements

21

$$\frac{1}{\Delta} = \left\{\frac{1}{2}A^{(i)} + \frac{e^2+e_j^2}{2}\left(-4i^2A^{(i)} + 2A_1^{(i)} + 2A_2^{(i)}\right) - \frac{1}{2}\eta^2B^{(i-1)}\right\}\cdot\cos i\,(l'-\lambda') +$$

$$+\frac{ee_j}{4}\left\{(4i+2i)\,A^{(i)} - 2A_1^{(i)} - 2A_2^{(i)}\right\}\cos\left[(i+1)\,l' - (i+1)\,\lambda' - \overline{\omega}' + \overline{\omega}\right] +$$

$$+\frac{e}{2}\left\{-2iA^{(i)} - A_1^{(i)}\right\}\cos\left[il' - (i-1)\,\lambda' - \overline{\omega}\right] +$$

$$+\frac{e_j}{2}\left\{(2i+1)\,A^{(i)} + A_1^{(i)}\right\}\cos\left[(i+1)\,l' - i\lambda' - \overline{\omega}'\right] +$$

$$+\frac{e^2}{2}\left\{4i^2-5i)\,A^{(i)} + (4i-2)\,A_1^{(i)} + 2A_2^{(i)}\right\}\cdot\cos\left[il' - (i-2)\,\lambda' - 2\overline{\omega}\right] +$$

$$+\frac{ee_j}{4}\left\{(-4i^2-2i)A^{(i)} + (-4i-2)A_1^{(i)} - 2A_2^{(i)}\right\}\cdot\cos\left[(i+1)\,l' - (i-1)\,\lambda' - \overline{\omega}' - \overline{\omega}\right] +$$

$$+\frac{e_j^2}{8}\left\{(4i^2+9i+4)\,A^{(i)} + (4i+6)\,A_1^{(i)} + 2A_2^{(i)}\right\}\cdot\cos\left[(i+2)\,l' - i\lambda' - 2\overline{\omega}'\right] +$$

$$+\frac{\eta^2}{2}B^{(i-1)}\cos\left[il' - (i-2)\,\lambda' - 2\tau'\right] + \ldots, \tag{2.40}$$

$$-\frac{r\cos\vartheta}{r_j^2} = \left\{-\frac{\alpha}{a_j} + \frac{1}{2}\frac{\alpha}{a_j}(e^2+e_j^2) + \frac{\alpha}{a_j}\eta^2\right\}\cos(l'-\lambda') -$$

$$-\frac{\alpha}{a_j}ee_j\cos(2l'-2\lambda'-\overline{\omega}'+\overline{\omega}) + \frac{3}{2}\frac{\alpha}{a_j}e\cos(l'-\overline{\omega}) -$$

$$-\frac{1}{2}\frac{\alpha}{a_j}e\cos(l'-2\lambda'+\overline{\omega}) - 2\frac{\alpha}{a_j}e_j\cos(2l'-\lambda'-\overline{\omega}') -$$

$$-\frac{1}{8}\frac{\alpha}{a_j}e^2\cos(l'+\lambda'-2\overline{\omega}') - \frac{3}{8}\frac{\alpha}{a_j}e^2\cos(l'-3\lambda'+2\overline{\omega}) +$$

$$+3\frac{\alpha}{a_j}ee_j\cos(2l'-\overline{\omega}'-\overline{\omega}) - \frac{1}{8}\frac{\alpha}{a_j}e_j^2\cos(l'+\lambda'-2\overline{\omega}') -$$

$$-\frac{27}{8}\frac{\alpha}{a_j}e_j^2\cos(3l'-\lambda'-2\overline{\omega}') - \frac{\alpha}{a_j}\eta^2\cos(l'+\lambda'-2\tau') + \ldots, \tag{2.41}$$

where the sign of summation over i is left out in (2.40), with i running through all integer values from $-\infty$ to $+\infty$.

In these expressions we have retained, with slight modifications, the notation used by Le Verrier:

e -- eccentricity of the orbit of the small planet

e_j -- eccentricity of the Jupiter orbit

$\eta = \sin(\gamma/2)$ -- sine of half the inclination of the plane of the orbit of the small planet to the plane of the Jupiter orbit,

$\alpha = a/a_j$ -- ratio of the major semi axes of the osculating orbit of the small planet and of Jupiter.

The angular variables have the following meaning:

$\lambda' = M + \omega + \Omega$ -- mean longitude of the small planet, measured first in the plane of the Jupiter orbit and then in the plane of the orbit of the small planet.

$\ell' = M_j + \pi_j$ (π_j -- longitude of the Jupiter perihelium) -- mean longitude of Jupiter in the orbit.

$\bar{\omega} = \omega + \Omega$ -- longitude of the perihelium of the small planet, measured in the same way as ',

$\bar{\omega}' = \pi_j$ -- longitude of the perihelium of Jupiter,

$\tau' = \Omega$ -- longitude of the node of the small planet, measured in the plane of the Jupiter orbit.

The geometric meanings of these quantities is clear from Fig. 1.

In this figure: [P] -- fictitious average small planet, [I] -- fictitious average Jupiter, II' -- plane of the Jupiter orbit, AA' -- osculating plane of the orbit of the small planet.

In this case

$$\lambda' = \overset{\frown}{E\Omega} + \overset{\frown}{\Omega[P]},\ l' = \overset{\frown}{E[J]},\ \tilde{\omega} = \overset{\frown}{E\Omega} + \overset{\frown}{\Omega\Pi},\ \tilde{\omega}' = \overset{\frown}{E\Pi_j},\ \tau' = \overset{\frown}{E\Omega}.$$

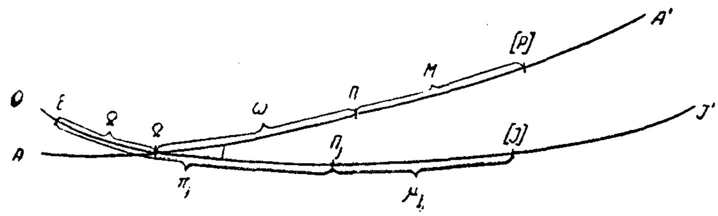

Fig. 1.

As regards the quantities $A^{(i)}$, $A_1^{(i)}$, $A_2^{(i)}$, and $B^{(i)}$, contained in (2.40), these are functions of a and α and can be expressed in terms of the Laplace coefficients $b^{(i)}_{n,0}$ and their derivatives $db^{(i)}_{n,0}/d\alpha$ in the following fashion [16]:

$$a_j A^{(i)} = b^{(i)}_{1,0},\ a_j A_1^{(i)} = b^{(i)}_{1,1},\ a_j A_2^{(i)} = \frac{1}{2} b^{(i)}_{1,2},\ a_j B^{(i)} = \alpha b^{(i)}_{3,0}. \qquad (2.42)$$

We put here

$$\alpha^m \frac{d^m b^{(i)}_{n,0}}{d\alpha^m} = b^{(i)}_{n,m}. \qquad (2.43)$$

The coefficients $b^{(i)}_{n,0}$ themselves are obtained as coefficients of the Fourier series

$$\{1+\alpha^2-2\alpha\cos(l'-\lambda')\}^{-\frac{n}{2}}=\frac{1}{2}\sum_{i=-\infty}^{+\infty} b_{n,0}^{(i)}\cos i(l'-\lambda'), \qquad (2.44)$$

and are functions of α only.

The expansions (2.40) and (2.41) converge absolutely under the following necessary conditions which, as was pointed out to me by V. F. Proskurin, are not sufficient conditions:

$\alpha < 1$; $e < 0.6627 \ldots$, $e_j < 0.6627 \ldots$, where 0.6627 ... is the known Laplace limit; a third condition is that η^2 must satisfy the condition mentioned in [15]. As regards the remaining variables, they can assume all real values. N. S. Samoylova-Yakhontova [17] has shown, for the plane case, that in the motion of Ceres the sufficient conditions for absolute convergence are also satisfied with respect to the major semi axes of Ceres and Jupiter and the eccentricities of their orbits.

In addition, we have written out the expansions (2.40) and (2.41) only to the second powers, inclusive, of the quantities e, e_j, and η. This is fully sufficient to explain all the features of the solution of the problem.

We now change over in the expansions (2.40) and (2.41) to our variables (2.32). We have the following formulas for the conversion

$$a=\frac{x_1^2}{k^2},\ e=\sqrt{2}\cdot\sqrt{\frac{x_2}{x_1}}\cdot\sqrt{1-\frac{x_2}{2x_1}};\ \eta^2=\frac{x_3}{2(x_1-x_2)},\ \lambda'=y_1+l',\ \bar{\omega}=l'-y_2,\ \tau'=l'-y_3. \tag{2.45}$$

It is seen from these formulas that y_1, y_2, and y_3 have the following geometrical meaning (Fig. 1)

$$y_2=\Omega\breve{[J]}-\Omega\breve{[J]},\ (2.46)\quad y_2=\Omega\breve{[J]}-\Omega\breve{[\mathrm{II}]},\ (2.47)\quad y_3=\Omega\breve{[J]}.\ (2.48)$$

Substituting (2.45) in (2.40) and (2.41) and confining ourselves to the second powers of $(x_2/x_1)^{1/2}$ and the first powers of $x_3/(x_1 - x_2)$ we obtain

$$W_j=\frac{1}{\Delta}-\frac{r\cos\vartheta}{r_j^2}, \tag{2.49}$$

$$\frac{1}{\Delta}=\sum_{i=-\infty}^{+\infty}[(1)^{(i)}+(2)^{(i)}+(3)^{(i)}+(4)^{(i)}+(5)^{(i)}+(6)^{(i)}+(7)^{(i)}+(8)^{(i)}], \tag{2.50}$$

$$-\frac{r\cos\vartheta}{r_j^2}=(11)+(21)+(31)+(32)+$$
$$+(41)+(51)+(52)+(61)+(71)+(72)+(81). \tag{2.51}$$

Here

$$\left.\begin{aligned}
(1)^{(i)}&=\Big\{\frac{1}{2}A^{(i)}+\frac{1}{8}\Big(\frac{2x_2}{x_1}+e_j^2\Big)\big(-4i^2A^{(i)}+2A_1^{(i)}+2A_2^{(i)}\big)-\\
&\quad-\frac{1}{2}\frac{x_3}{2(x_1-x_2)}\cdot B^{(i-1)}\Big\}\cos iy_1,\\
(2)^{(i)}&=\frac{1}{4}e_j\sqrt{\frac{2x_2}{x_1}}\{(4i^2+2i)A^{(i)}-2A_1^{(i)}-2A_2^{(i)}\}\times\\
&\quad\times\cos[(i+1)y_1+y_2-M_j],\\
(3)^{(i)}&=\frac{1}{2}\sqrt{\frac{2x_2}{x_1}}\{-2iA^{(i)}-A_1^{(i)}\}\cos[(i-1)\cdot y_1+y_2],\\
(4)^{(i)}&=\frac{1}{2}e_j\{[(2i+1)A^{(i)}+A_1^{(i)}\}\cos[iy_1-M_j],\\
(5)^{(i)}&=\frac{1}{8}\frac{2x_2}{x_1}\{(4i^2-5i)A^{(i)}+(4i-2)A_1^{(i)}+2A_2^{(i)}\}\times\\
&\quad\times\cos[(i-2)y_1-2y_2],
\end{aligned}\right\}(2.52)$$

$$(6)^{(i)} = \frac{1}{4} e_j \sqrt{\frac{2x_2}{x_1}} \{(-4i^2 - 2i) A^{(i)} + (-4i - 2) A_1^{(i)} - A_2^{(i)}\} \times$$
$$\times \cos [(i-1) y_1 - y_2 - M_j],$$
$$(7)^{(i)} = \frac{1}{8} e_j^2 \{(4i^2 + 9i + 4) A^{(i)} + (4i + 6) A_1^{(i)} + 2A_2^{(i)}\} \times$$
$$\times \cos [iy_1 - 2M_j],$$
$$(8)^{(i)} = \frac{1}{2} \frac{x_3}{2(x_1 - x_2)} \cdot B^{(i-1)} \cos [(i-2) y_1 - 2y_3];$$

$$\left.\begin{aligned}
(11) &= \left\{-\frac{\alpha}{a_j} + \frac{1}{2}\frac{\alpha}{a_j}\left(\frac{2x_2}{x_1} + e_j^2\right) + \frac{\alpha}{a_j}\cdot\frac{x_3}{2(x_1 - x_2)}\right\}\cos y_1,\\
(21) &= -\frac{\alpha}{a_j} e_j \sqrt{\frac{2x_2}{x_1}} \cos (2y_1 + y_2 - M_j),\\
(31) &= +\frac{3}{2}\frac{\alpha}{a_j}\sqrt{\frac{2x_2}{x_1}}\cdot\cos y_2,\ (32) = -\frac{1}{2}\frac{\alpha}{a_j}\sqrt{\frac{2x_2}{x_1}}\cos (2y_1 + y_2),\\
(41) &= -2\frac{\alpha}{a_j} e_j \cos (y_1 - M_j),\ (51) = -\frac{1}{8}\frac{\alpha}{a_j}\cdot\frac{2x_2}{x_1}\cdot\cos (y_1 + 2y_2),\\
(52) &= -\frac{3}{8}\frac{\alpha}{a_j}\cdot\frac{2x_2}{x_1}\cdot\cos (3y_1 + 2y_2),\ (61) = -3\frac{\alpha}{a_j} e_j \sqrt{\frac{2x_2}{x_1}}\cos (y_2 + M_j),\\
(71) &= -\frac{1}{8}\frac{\alpha}{a_j} e_j^2 \cos (y_1 + 2M_j),\ (72) = -\frac{27}{8}\frac{\alpha}{a_j} e_j^2 \cos (y_1 - 2M_j),\\
(81) &= -\frac{\alpha}{a_j}\cdot\frac{x_3}{2(x_1 - x_2)}\cdot\cos (y_1 + 2y_3).
\end{aligned}\right\} \quad (2.53)$$

$$\alpha = \frac{x_1^2}{a_j \cdot k^2}. \quad (2.54)$$

These expansions will converge for all values of x_1, x_2, x_3; y_1, y_2, y_3; M_j, e_j, and α provided the following necessary conditions are satisfied

$$\left.\begin{aligned}
\alpha = \frac{x_1^2}{a_j k^2} < 1,\ \sqrt{2}\sqrt{\frac{x_2}{x_1}}\sqrt{1 - \frac{x_2}{2x_1}} < 0.6627\ldots,\\
e_j < 0.6627\ldots,
\end{aligned}\right\} \quad (2.55)$$

$x_3/2(x_1 - x_2)$ satisfies the condition noted in [15], and the sufficient conditions derived in [17].

Let us note certain features of these expansions, which we shall use later on.

First, the argument of the cosine of the term independent of x_2, x_3, and e_j depends only on y_1.

Second, the arguments of the cosines of the terms independent of x_2 and x_3, but dependent on the powers of e_j (in our expansion these will be the terms (4), (7), (41), (71), and (72)), depend only on y_1 and M_j.

Third, only the terms containing x_3 depend on y_3. These are terms (8) and (81).

Fourth, W_j will depend on the time only through M_j, and the time dependent terms of the expansions (2.52) and (2.53) will contain e_j as a factor.

We know that the expansion of W_j in the variables used by Le Verrier converge absolutely to W_j within the limits indicated above. But since the formulas of the conversion (2.35) from these variables to the variables (2.32) have no analytic singularities whatever in the domain (2.55), the expansion W_j which we have obtained in the variables (2.32) will also converge absolutely to W_j.

5. Averaging the Characteristic Function Over x_2 and x_3

The characteristic function of our problem is determined by the formula

$$\Omega = \frac{k^4}{2x_1^2} + n_j(x_1 - x_2 - x_3) + k^2 m_j W_j. \qquad (2.56)$$

Let us change the variables x_1, y_1, and y_2 in (2.56) in accordance with the formulas

$$x_1 = \frac{1}{l_1}(\lambda - l_2 x_2 - l_3 x_3),\ y_1 = \frac{1}{m_1^{(1)}}(\mu_1 - m_3^{(1)} y_3),\ y_2 = \frac{1}{m_2^{(2)}}(\mu_2 - m_3^{(2)} y_3). \qquad (2.57)$$

Without loss of generality, as will be shown in the next chapter, we can put

$$l_1 = m_1^{(1)} = m_2^{(2)} = 1. \qquad (2.58)$$

Therefore

$$x_1 = \lambda - l_2 x_2 - l_3 x_3,\ y_1 = \mu_1 - m_3^{(1)} y_3,\ y_2 = \mu_2 - m_3^{(2)} y_3. \qquad (2.59)$$

We must substitute the expressions of (2.59) into (2.56), and assume λ, μ_1, and μ_2 constant during the averaging. We can then write out the average values of the terms of the right half of (2.56). For this purpose we determine, after processing the observational material, the average values of x_2 and x_3. We denote them by x_{20}

and x_{30}. In addition, we can find also the upper and lower limits of the deviations for these quantities. We denote them respectively by $\bar{x}_2$, $\bar{x}_3$ and $\underline{x}_2$, $\underline{x}_3$, and generally speaking

$$x_{20} \neq \frac{\bar{x}_2 + \underline{x}_1}{2}, \quad x_{30} \neq \frac{\bar{x}_3 + \underline{x}_3}{2}. \tag{2.60}$$

We now calculate the average value of the first term in the right half of (2.56), where x_1 is replaced in accordance with (2.59).

We have

$$\frac{k^4}{2x_1^2} = \frac{k^4}{2x_0^2} + \frac{\partial}{\partial x_2}\left(\frac{k^4}{2x_1^2}\right)\Bigg|_{\substack{x_2=x_{20}\\ x_3=x_{30}}} \Delta x_2 + \frac{\partial}{\partial x_3}\left(\frac{k^4}{2x_1^2}\right)\Bigg|_{\substack{x_2=x_{20}\\ x_3=x_{30}}} \cdot \Delta x_3 + \dots, \tag{2.61}$$

where

$$x_{10} = \lambda - l_2 x_{20} - l_3 x_{30}, \quad \Delta x_2 = x_2 - x_{20}, \quad \Delta x_3 = x_3 - x_{30}, \tag{2.62}$$

and

$$\frac{\partial}{\partial x_2}\left(\frac{k^4}{2x_1^2}\right)\Bigg|_{\substack{x_2=x_{20}\\ x_3=x_{30}}} = \frac{d}{dx_1}\left(\frac{k^4}{2x_1^2}\right)\cdot\frac{\partial x_1}{\partial x_2}\Bigg|_{\substack{x_2=x_{20}\\ x_3=x_{30}}} = -\frac{k^4}{x_{10}^3}(-l_2), \quad \frac{\partial}{\partial x_3}\left(\frac{k^4}{2x_1^2}\right)\Bigg|_{\substack{x_2=x_{20}\\ x_3=x_{30}}} =$$

$$= -\frac{k^4}{x_{10}^3}(-l_3), \dots \tag{2.63}$$

Therefore the average value of $k^4/2x_1^2$ is

$$\frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int\limits_{x_2=\underline{x}_2}^{\overline{x}_2} \int\limits_{x_3=\underline{x}_3}^{\overline{x}_3} \int\limits_{y_3=\underline{y}_3}^{\overline{y}_3} \frac{k^4}{2x_1^2} \cdot dx_2\, dy_2\, dy_3 = \frac{k^4}{2x_{10}^2} + \frac{k^4}{x_{10}^3} \cdot l_2 \delta x_2 +$$

$$+ \frac{k^4}{x_{10}^3} \cdot l_3 \delta x_3 + \ldots, \tag{2.64}$$

where

$$\delta x_2 = \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int\limits_{x_2=\underline{x}_2}^{\overline{x}_2} \int\limits_{x_3=\underline{x}_3}^{\overline{x}_3} \int\limits_{y_3=\underline{y}_3}^{\overline{y}_3} \Delta x_2\, dx_2\, dx_3\, dy_3 =$$

$$= \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \int\limits_{x_2=\underline{x}_2}^{\overline{x}_2} \Delta x_2\, dx_2 = \frac{1}{2} [(\overline{x}_2 - x_{20}) - (x_{20} - \underline{x}_2)], \tag{2.65}$$

and analogously

$$\delta x_3 = \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int\limits_{x_2=\underline{x}_2}^{\overline{x}_2} \int\limits_{x_3=\underline{x}_3}^{\overline{x}_3} \int\limits_{y_3=\underline{y}_3}^{\overline{y}_3} \Delta x_3\, dx_2\, dx_3\, dy_3 =$$

$$= \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \int\limits_{x_3=\underline{x}_3}^{\overline{x}_3} \Delta x_3\, dx_3 = \frac{1}{2} [(\overline{x}_3 - x_{30}) - (x_{30} - \underline{x}_3)]. \tag{2.66}$$

We now determine the second term of the characteristic function $n_j(x_1 - x_2 - x_3)$, where $x_1 = \lambda - l_2 x_2 - l_3 x_3$. We obtain

$$\frac{1}{\overline{x}_2-\underline{x}_2}\cdot\frac{1}{\overline{x}_3-\underline{x}_3}\cdot\frac{1}{\overline{y}_3-\underline{y}_3}\cdot\int\limits_{x_2=\underline{x}_2}^{\overline{x}_2}\int\limits_{x_3=\underline{x}_3}^{\overline{x}_3}\int\limits_{y_3=\underline{y}_3}^{\overline{y}_3} n_j(x_1-x_2-x_3)\,dx_2\,dx_3\,dy_3 =$$

$$= n_j(x_{10}-x_{20}-x_{30})-(1+l_2)\,n_j\,\delta x_2-(1+l_3)\,n_j\,\delta x_3+\ldots \qquad (2.67)$$

Here

$$x_{10}=\lambda-l_2x_{20}-l_3x_{30}, \qquad (2.68)$$

and δx_2 and δx_3 are determined from (2.65) and (2.66).

We now average the term $k^2m_jW_j$.

In the explicit expression for W_j, given by formulas (2.50), (2.51), (2.52), (2.53), and (2.54) we must replace x_1, y_1, and y_2 in accord with (2.59). Then the general term W_j assumes the form

$$(n)^{(i)}=K_n^{(i)}\cos\left(M_n^{(i)}+\varkappa_n^{(i)}y_3'\right), \qquad (2.69)$$

where

$$M_n^{(i)}=m_{1n}^{(i)}\mu_1+m_{2n}^{(i)}\mu_2+m_{3n}^{(i)}M_j, \qquad (2.70)$$

$$y_3'=m_3^{(1)}\cdot y_3, \qquad (2.71)$$

and the quantities $K_n^{(i)}$ are independent of the variables contained in the arguments of the cosine, but do depend on λ, x_2, and x_3. As regards the argument of the cosine, it does not depend on λ, x_2, or x_3. Therefore

$$\frac{1}{\overline{x}_2-\underline{x}_2}\cdot\frac{1}{\overline{x}_3-\underline{x}_3}\cdot\frac{1}{\overline{y}_3-\underline{y}_3}\cdot\int\limits_{x_2=\underline{x}_2}^{\overline{x}_2}\int\limits_{x_3=\underline{x}_3}^{\overline{x}_3}\int\limits_{y_3=\underline{y}_3}^{\overline{y}_3}(n)^{(i)}\,dx_2\,dx_3\,dy_3=$$

$$=\overset{\cong}{(n)}{}^{(i)}=\frac{1}{\overline{x}_2-\underline{x}_2}\cdot\frac{1}{\overline{x}_3-\underline{x}_3}\cdot\int\limits_{x_2=\underline{x}_2}^{\overline{x}_2}\int\limits_{x_3=\underline{x}_2}^{\overline{x}_3}K_n^{(i)}\,dx_2dx_3\,\frac{1}{\overline{y}_3-\underline{y}_3}\cdot\int\limits_{y_3=\underline{y}_3}^{\overline{y}_3}\cos\left(M_n^{(i)}+\varkappa_n^{(i)}y_3'\right)dy_3. \tag{2.72}$$

In the present section we calculate

$$\overline{\overline{K}}{}_n^{(i)}=\frac{1}{\overline{x}_2-\underline{x}_2}\cdot\frac{1}{\overline{x}_3-\underline{x}_3}\cdot\int\limits_{x_2=\underline{x}_2}^{\overline{x}_2}\int\limits_{x_3=\underline{x}_3}^{\overline{x}_3}K_n^{(i)}\,dx_2\;dx_3. \tag{2.73}$$

Let us expand $K_n^{(i)}$ in powers of Δx_2 and Δx_3:

$$K_n^{(i)}=K_{n0}^{(i)}+\left.\frac{\partial K_n^{(i)}}{\partial x_2}\right|_{\substack{x_2=x_{20}\\x_3=x_{30}}}\cdot\Delta x_2+\left.\frac{\partial K_n^{(i)}}{\partial x_3}\right|_{\substack{x_2=x_{20}\\x_3=x_{30}}}\cdot\Delta x_3+\ldots, \tag{2.74}$$

where $K_{n0}^{(i)}$ is the value of $K_n^{(i)}$ when $x_2 = x_{20}$ and $x_3 = x_{30}$.

Therefore

$$\overline{\overline{K}}{}_n^{(i)}=K_{n0}^{(i)}+\left.\frac{\partial K_n^{(i)}}{\partial x_2}\right|_{\substack{x_2=x_{20}\\x_3=x_{30}}}\cdot\delta x_2+\left.\frac{\partial K_n^{(i)}}{\partial x_3}\right|_{\substack{x_2=x_{20}\\x_3=x_{30}}}\cdot\delta x_3+\ldots \tag{2.75}$$

These operations are valid, since we average in those domains (2.55) where the expansions (2.50), (2.51), (2.52), (2.53), and (2.54) converge absolutely.

The specific expressions for $K_{n0}^{(i)}$, $\partial K_n^{(i)}/\partial x_2$, and

$\partial K_n^{(i)}/\partial x_3$ will be derived in Chapter VII.

The values of δx_2 and δx_3 are determined from (2.65) and (2.66).

6. Averaging of W_j Over y_3

The averaging of $(n)^{(i)}$ over y_3 reduces to a calculation, as was already noted, of the quantities

$$\frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int\limits_{y_3=\underline{y}_3}^{\overline{y}_3} \cos\left[M_n^{(i)} + \varkappa_n^{(i)} y_3'\right] dy_3, \tag{2.76}$$

where

$$y_3' = m_3^{(1)} y_3. \tag{2.71}$$

Therefore

$$\frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int\limits_{y_3=\underline{y}_3}^{\overline{y}_3} \cos\left[M_n^{(i)} + \varkappa_n^{(i)} y_3'\right] dy_3 = \frac{1}{\varkappa_n^{(i)}\left(\overline{y}_3' - \underline{y}_3'\right)} \left\{\sin\left[M_n^{(i)} + \varkappa_n^{(i)} \overline{y}_3'\right] - \right.$$

$$\left. - \sin\left[M_n^{(i)} + \varkappa_n^{(i)} \underline{y}_3'\right]\right\}, \tag{2.77}$$

where

$$\overline{y}_3' = m_3^{(1)} \overline{y}_3, \; \underline{y}_3' = m_3^{(1)} \underline{y}_3. \tag{2.78}$$

Thus, actually the averaging over y_3 can be replaced by averaging over y_3'.

Let us prove now that after adding the terms $\overset{\approx}{(n)}{}^{(i)}$ we obtain the average value of the perturbation function

$\overset{\approx}{W}_j$. The correctness of this statement follows from the following remarks.

Remark 1. The function W_j, represented by the series (2.52) and (2.53), is an analytic function in its variables x_i and y_i ($i = 1, 2, 3$) in the domain (2.55) which contains, together with its finite vicinity, the subspace $x_i = 0$ ($i = 1, 2, 3$) of the complex space x_i, y_i ($i = 1, 2, 3$).

Remark 2. Consequently the following operation is valid (if we substitute for x_1 in accord with (2.57)):

$$\frac{1}{\overline{x}_2-\underline{x}_2}\cdot\frac{1}{\overline{x}_3-\underline{x}_3}\cdot\frac{1}{\overline{y}_3-\underline{y}_3}\cdot\int\limits_{x_2=\underline{x}_2}^{\overline{x}_2}\int\limits_{x_3=\underline{x}_3}^{\overline{x}_3}\int\limits_{y_3=\underline{y}_3}^{\overline{y}_3} W_j\,dx_2\,dx_3\,dy_3 = \overset{\approx}{W}_j =$$

$$= \sum_{n,\,i}\frac{1}{\overline{x}_2-\underline{x}_2}\cdot\frac{1}{\overline{x}_3-\underline{x}_3}\cdot\frac{1}{\overline{y}_3-\underline{y}_3}\cdot\int\limits_{x_2=\underline{x}_2}^{\overline{x}_2}\int\limits_{x_3=\underline{x}_3}^{\overline{x}_3}\int\limits_{y_3=\underline{y}_3}^{\overline{y}_3}(n)^{(i)}\,dx_2\,dx_3\,dy_3 = \sum_{n,\,i}\overset{\approx}{(n)}^{(i)}. \qquad (2.79)$$

Remark 3. The function W_j is an almost-periodic function of y_3' in the sense of Bohl [18] after substituting for y_1 and y_2 in accord with (2.57) with basis 1, $m_3^{(2)}/m_3^{(1)}$, and $1/m_3^{(1)}$.

Actually, the argument of the cosine of each term W_j contains y_1, y_2, and y_3 in a linear combination with integral coefficients, on the one hand. On the other hand, generally speaking, the quantities $m_3^{(1)}$, $m_3^{(2)}$, and $m_3^{(2)}/m_3^{(1)}$ are irrational. But after replacing y_1 by

$\mu_1 - m_3^{(1)} y_3$ and y_2 by $\mu_2 - m_3^{(2)} y_3$, the argument of the cosine is a linear combination of the quantities $m_3^{(1)} y_3$, $m_3^{(2)} y_3$, y_3 with integral coefficients, i.e., W_j will be, by virtue of the uniqueness of the expansion, an almost-periodic function of y_3 with basis $m_3^{(1)}$, $m_3^{(2)}$, 1. With respect to the quantity y_3', on the other hand, W_j will be an almost-periodic function with bases 1, $m_3^{(2)}/m_3^{(1)}$, $1/m_3^{(1)}$.

Remark 4. By virtue of the almost-periodic behavior of W_j with respect to y_3', we have, in accordance with the preceding remark

$$\overline{\overline{\widetilde{W}}}_j = \sum_{n,\, i} \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \int_{x_2 = \underline{x}_2}^{\overline{x}_2} \int_{x_3 = \underline{x}_3}^{\overline{x}_3} K_n^{(i)}\, dx_2\, dx_3 \frac{1}{\overline{y}_3' - \underline{y}_3'} \int_{y_3' = \underline{y}_3'}^{\overline{y}_3'} \cos\,[M_n^{(i)} + x_n^{(i)} y_3']\, dy_3'. \tag{2.80}$$

These expressions will be calculated in detail in Chapter VII.

7. Determination of the Terms of the Series Representing the Solutions

Following the foregoing averaging, the characteristic function will depend on x_i and y_i $(i = 1, 2, 3)$ only through λ, μ_1, and μ_2.

We now assume that the coefficients ℓ_2, ℓ_3, $m_3^{(1)}$, and

$m_3^{(2)}$ are such that the orthointerpolational conditions

$$1 + l_3 m_3^{(1)} = 0, \quad l_2 + m_3^{(2)} l_3 = 0, \tag{2.81}$$

are satisfied, where we put $l_1 = m_1^{(1)} = m_2^{(2)} = 1.$

In this case, as we already know (see Section 2 of the present chapter), the solutions of the system of differential equations in which the partial derivatives of the characteristic function Ω are replaced by the derivatives of the function

$$\widetilde{\Omega} = \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int\limits_{x_2 = \underline{x}_2}^{\overline{x}_2} \int\limits_{x_3 = \underline{x}_3}^{\overline{x}_3} \int\limits_{y_3 = \underline{y}_3}^{\overline{y}_3} \Omega^* dx_2\, dx_3\, dy_3, \tag{2.82}$$

are determined by the equations

$$\left.\begin{aligned} x_1 &= x_{1n} + \frac{\partial \Omega}{\partial \mu_1}, \\ x_2 &= x_{2n} + \frac{\partial \Omega}{\partial \mu_2}, \\ x_3 &= x_{3n} + m_3^{(1)} \frac{\partial \Omega}{\partial \mu_1} + m_3^{(2)} \frac{\partial \Omega}{\partial \mu_2}, \end{aligned}\right\} \tag{2.83}$$

$$\left.\begin{aligned} y_1 &= y_{1n} - \frac{\partial \Omega}{\partial \lambda}, \\ y_2 &= y_{2n} - l_2 \frac{\partial \Omega}{\partial \lambda}, \\ y_3 &= y_{3n} - l_3 \frac{\partial \Omega}{\partial \lambda}, \end{aligned}\right\} \tag{2.84}$$

where

$$\Omega = \int\limits_{t = t_0}^{t} \widetilde{\Omega}\, dt, \tag{2.85}$$

and x_{1n}, x_{2n}, x_{3n}; y_{1n}, y_{2n}, y_{3n} are the initial values of our variables (2.32) for $t = t_0$.

In the case (2.81) the formulas (2.83), (2.84), and (2.85) determine the following plan of action towards obtaining the solutions (2.83) and (2.84).

It is necessary to integrate our characteristic function $\tilde{\Omega}$ with respect to t and obtain $\underline{\Omega}$.

The function $\underline{\Omega}$ must be differentiated with respect to λ, μ_1, and μ_2 and substituted in (2.83) and (2.84).

As regards the determination of the initial values of our variables, this question will be considered in the following chapters.

We carry out successively the first two operations.

1. Determination of $\underline{\Omega}$

The time t is contained in the function $\tilde{\Omega}$ only through M_j:

$$M_j = M_{j0} + n_j (t - t_0). \tag{2.86}$$

On the other hand, the quantity M_j, as was already shown, is contained in $\tilde{\Omega}$ only through the function W_j, and it is seen from an examination of the expansion of W_j, as we have noted in Section 4, that the members of W_j containing M_j must contain also e_j as a factor. Therefore when $e_j = 0$ (the case of circular bounded three-point problem) the characteristic function $\underline{\Omega}$, just like $\tilde{\Omega}$,

is independent of t.

We denote the totality of the members in $\widetilde{\Omega}$ which are independent of t by $\widetilde{\Omega}_0$. These include the following: 1) The average values of the members $k_4/2x_1^2$ and $n_j(x_1 - x_2 - x_3)$, 2) the averaged values of those terms of the function W_j, which do not contain e_j explicitly. This contributes to the function $\underline{\Omega}$ a member $\widetilde{\Omega}_0(t - t_0)$, which we shall call the secular term of the function $\underline{\Omega}$.

As regards the members of W_j, containing M_j, these will contribute to $\underline{\Omega}$ periodic members with period $2\pi/n_j$ with respect to t, since the M_j are contained in the argument of the cosine both before and after the averaging, except that the coefficients are integral, as can be seen from formulas (2.52) and (2.53).

Thus, if the general term of the averaged function $\widetilde{\overline{\overline{W}}}_j$ is

$$\widetilde{\overline{\overline{(n)}}}^{(i)} = \frac{1}{\varkappa_n^{(i)}(\overline{y}_3' - \underline{y}_3')} \cdot \overline{\overline{K}}_n^{(i)} [\sin (M_n^{(i)} + \varkappa_n^{(i)} \overline{y}_3') - \sin (M_n^{(i)} + \varkappa_n^{(i)} \underline{y}_3')], \qquad (2.87)$$

where

$$\overline{\overline{K}}_n^{(i)} = \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \int_{x_2 = \underline{x}_2}^{\overline{x}_2} \int_{x_3 = \underline{x}_3}^{\overline{x}_3} K_n^{(i)} \, dx_2 \, dx_3, \qquad (2.88)$$

$$M_n^{(i)} = m_{1n}^{(i)} \mu_1 + m_{2n}^{(i)} \mu_2 + m_{3n}^{(i)} \cdot M_j, \qquad (2.70)$$

then in the case when $m_{3n}^{(i)} = 0$ we obtain precisely the secular terms in Ω. On the other hand, if $m_{3n}^{(i)} \neq 0$, then

$$\{n\}^{(i)} = \int_{t=t_0}^{t} \overline{\overline{(n)}}^{(i)} dt = \frac{1}{\varkappa_n^{(i)}(\overline{y}_3' - \underline{y}_3') m_{3n}^{(i)} \cdot n_j} \cdot \overline{\overline{K}}_n^{(i)} \cdot [-\cos(M_n^{(i)} + \varkappa_n^{(i)} \overline{y}_3') +$$
$$+ \cos(M_{n0}^{(i)} + \varkappa_n^{(i)} \overline{y}_3') + \cos(M_n^{(i)} + \varkappa_n^{(i)} \underline{y}_3') - \cos(M_{n0}^{(i)} + \varkappa_n^{(i)} \underline{y}_3')]. \tag{2.89}$$

where

$$M_{n0}^{(i)} = m_{1n}^{(i)} \mu_1 + m_{2n}^{(i)} \mu_2 + m_{3n}^{(i)} M_{j0}. \tag{2.90}$$

2. Derivation of $\frac{\partial \Omega}{\partial \lambda}$, $\frac{\partial \Omega}{\partial \mu_1}$, $\frac{\partial \Omega}{\partial \mu_2}$

a) Derivation of $\partial \Omega / \partial \lambda$.

Differentiating the first two terms of the function Ω successively with respect to λ, we obtain

$$\frac{\partial}{\partial \lambda} \left\{ \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int_{x_2 = \underline{x}_2}^{\overline{x}_2} \int_{x_3 = \underline{x}_3}^{\overline{x}_3} \int_{y_3 = \underline{y}_3}^{\overline{y}_3} \frac{k^4}{2x_1^2} \cdot dx_2\, dx_3\, dy_3 \cdot (t - t_0) \right\} =$$
$$= \left\{ -\frac{k^4}{x_{10}^3} - \frac{3k^4}{x_{10}^4} \cdot l_2 \delta x_2 - \frac{3k^4}{x_{10}^4} \cdot l_3 \delta x_3 - \dots \right\} \cdot (t - t_0), \tag{2.91}$$

$$\frac{\partial}{\partial \lambda} \left\{ \frac{1}{\overline{x}_2 - \underline{x}_2} \cdot \frac{1}{\overline{x}_3 - \underline{x}_3} \cdot \frac{1}{\overline{y}_3 - \underline{y}_3} \cdot \int_{x_2 = \underline{x}_2}^{\overline{x}_2} \int_{x_3 = \underline{x}_3}^{\overline{x}_3} \int_{y_3 = \underline{y}_3}^{\overline{y}_3} n_j (x_1 - x_2 - x_3)\, dx_2\, dx_3\, dy_3 \,(t - t_0) \right\} =$$
$$= n_j (t - t_0). \tag{2.92}$$

Let us now differentiate the function W_j with respect to λ.

If we use the explicit expressions (2.52) and (2.53) for W_j and take (2.87), (2.88), (2.70), (2.89), and (2.90)

into account, then in the case when $m_{3n}^{(i)} = 0$ we obtain

$$\frac{\partial}{\partial\lambda}\{n\}^{(i)} = \frac{t-t_0}{\varkappa_n^{(i)}(\bar{y}_3'-\underline{y}_3')} \cdot \frac{\partial \bar{\bar{K}}_n^{(i)}}{\partial\lambda}[\sin(M_n^{(i)}+\varkappa_n^{(i)}\bar{y}_3') - \sin(M_n^{(i)}+\varkappa_n^{(i)}\underline{y}_3')], \quad (2.93)$$

since only $K_n^{(i)}$ depends on λ.

On the other hand, if $m_{3n}^{(i)} \neq 0$ we obtain

$$\frac{\partial}{\partial\lambda}\{n\}^{(i)} = \frac{1}{\varkappa_n^{(i)}(\bar{y}_3'-\underline{y}_3')\, m_{3n}^{(i)} n_j} \cdot \frac{\partial \bar{\bar{K}}_n^{(i)}}{\partial\lambda} \times [-\cos(M_n^{(i)}+\varkappa_n^{(i)}\bar{y}_3') +$$

$$+\cos(M_{no}^{(i)}+\varkappa_n^{(i)}\bar{y}_3') + \cos(M_n^{(i)}+\varkappa_n^{(i)}\underline{y}_3') - \cos(M_{no}^{(i)}+\varkappa_n^{(i)}\underline{y}_3')]. \quad (2.94)$$

After multiplying by $k^2 m_j$ we obtain an expression also for the third term $\partial\Omega/\partial\lambda$.

b) Derivation of $\partial\Omega/\partial\mu_1$, $\partial\Omega/\partial\mu_2$

First,

$$\left.\begin{aligned}
&\frac{\partial}{\partial\mu_1}\left[\frac{1}{\bar{x}_2-\underline{x}_2}\cdot\frac{1}{\bar{x}_3-\underline{x}_3}\cdot\frac{1}{\bar{y}_3-\underline{y}_3}\cdot\int_{x_2=\underline{x}_2}^{\bar{x}_2}\int_{x_3=\underline{x}_3}^{\bar{x}_3}\int_{y_3=\underline{y}_3}^{\bar{y}_3}\frac{k_4}{2x_1^2}\,dx_2\,dx_3\,dy_3\,(t-t_0)\right] = \\
&= \frac{\partial}{\partial\mu_2}\left[\frac{1}{\bar{x}_2-\underline{x}_2}\cdot\frac{1}{\bar{x}_3-\underline{x}_3}\cdot\frac{1}{\bar{y}_3-\underline{y}_3}\cdot\int_{x_2=\underline{x}_2}^{\bar{x}_2}\int_{x_3=\underline{x}_3}^{\bar{x}_3}\int_{y_3=\underline{y}_3}^{\bar{y}_3}\frac{k_4}{2x_1^2}\times\right. \\
&\qquad\left.\times\,dx_2\,dx_3\,dy_3\,(t-t_0)\right] = 0, \\
&\frac{\partial}{\partial\mu_1}\left[\frac{1}{\bar{x}_2-\underline{x}_2}\cdot\frac{1}{\bar{x}_3-\underline{x}_3}\cdot\frac{1}{\bar{y}_3-\underline{y}_3}\cdot\int_{x_2=\underline{x}_2}^{\bar{x}_2}\int_{x_3=\underline{x}_3}^{\bar{x}_3}\int_{y_3=\underline{y}_3}^{\bar{y}_3}\times\right. \\
&\qquad\left.\times\,n_j(x_1-x_2-x_3)\,dx_2\,dx_3\,dy_3\,(t-t_0)\right] = \\
&= \frac{\partial}{\partial\mu_2}\left[\frac{1}{\bar{x}_2-\underline{x}_2}\cdot\frac{1}{\bar{x}_3-\underline{x}_3}\cdot\frac{1}{\bar{y}_3-\underline{y}_3}\cdot\int_{x_2=\underline{x}_2}^{\bar{x}_2}\int_{x_3=\underline{x}_3}^{\bar{x}_3}\int_{y_3=\underline{y}_3}^{\bar{y}_3}\times\right. \\
&\qquad\left.\times\,n_j(x_1-x_2-x_3)\,dx_2\,dx_3\,dy_3\,(t-t_0)\right] = 0,
\end{aligned}\right\} \quad (2.95)$$

A1

because the first two terms of Ω are independent of μ_1 and μ_2.

Second, taking into consideration (2.87), (2.88), (2.70), (2.89), and (2.90), we obtain the following values for the members of the expansion of the third terms of $\partial\Omega/\partial\mu_1$ and $\partial\Omega/\partial\mu_2$.

If $m_{3n}^{(i)} = 0$, then

$$\frac{\partial}{\partial\mu_1}\{n\}^{(i)} = \frac{m_{1n}^{(i)}}{(\overline{y}_3' - \underline{y}_3')\varkappa_n^{(i)}} \cdot \overline{\overline{K}}_n^{(i)} [\cos(M_n^{(i)} + \varkappa_n^{(i)}\overline{y}_3') - \cos(M_n^{(i)} + \varkappa_n^{(i)}\underline{y}_3')] \cdot (t - t_0), \tag{2.96}$$

$$\frac{\partial}{\partial\mu_2}\{n\}^{(i)} = \frac{m_{2n}^{(i)}}{(\overline{y}_3' - \underline{y}_3')\varkappa_n^{(i)}} \cdot \overline{\overline{K}}_n^{(i)} [\cos(M_n^{(i)} + \varkappa_n^{(i)}\overline{y}_3') - \cos(M_n^{(i)} + \varkappa_n^{(i)}\underline{y}_3')] \cdot (t - t_0). \tag{2.97}$$

On the other hand, if $m_{3n}^{(i)} \neq 0$, we obtain

$$\frac{\partial}{\partial\mu_1}\{n\}^{(i)} = \frac{m_{1n}^{(i)}}{(\overline{y}_3' - \underline{y}_3')\varkappa_n^{(i)} m_{3n}^{(i)} n_j} \cdot \overline{\overline{K}}_n^{(i)} \times [\sin(M_n^{(i)} + \varkappa_n^{(i)}\overline{y}_3') - \sin(M_{n0}^{(i)} + \varkappa_n^{(i)}\overline{y}_3') - \sin(M_n^{(i)} + \varkappa_n^{(i)}\underline{y}_3') + \sin(M_{n0}^{(i)} + \varkappa_n^{(i)}\underline{y}_3')], \tag{2.98}$$

$$\frac{\partial}{\partial\mu_2}\{n\}^{(i)} = \frac{m_{2n}^{(i)}}{(\overline{y}_3' - \underline{y}_3')\varkappa_n^{(i)} m_{3n}^{(i)} n_j} \cdot \overline{\overline{K}}_n^{(i)} \times [\sin(M_n^{(i)} + \varkappa_n^{(i)}\overline{y}_3') - \sin(M_{n0}^{(i)} + \varkappa_n^{(i)}\overline{y}_3') - \sin(M_n^{(i)} + \varkappa_n^{(i)}\underline{y}_3') + \sin(M_{n0}^{(i)} + \varkappa_n^{(i)}\underline{y}_3')]. \tag{2.99}$$

After multiplying expressions (2.96) and (2.98) by $k^2 m_j$ and summing over n and i, we obtain an expression also for the third term of $\partial\Omega/\partial\mu_1$. We obtain $\partial\Omega/\partial\mu_2$ analogously. Expressions (2.96) and (2.97) will determine the secular terms in $\partial\Omega/\partial\mu_1$ and $\partial\Omega/\partial\mu_2$.

The solutions of the system of differential equations itself, after calculating $\partial\Omega/\partial\lambda$, $\partial\Omega/\partial\mu_1$, and $\partial\Omega/\partial\mu_2$, are determined by formulas (2.83) and (2.84).

8. Certain Properties of the Solutions of the Differential Equations of Triple-averaged Ortho-interpolational Bounded Three-dimensional Elliptical Three-point Problem

As was already noted, the characteristic function $\tilde{\Omega}$ is such that 1) the totality of the terms containing the time explicitly has e_j as a factor and 2) the third term of the characteristic function $\tilde{\Omega}$ has $k^2 m_j$ as a factor. The first property will also be possessed by the functions Ω, $\frac{\partial\Omega}{\partial\lambda}$, $\frac{\partial\Omega}{\partial\mu_1}$ and $\frac{\partial\Omega}{\partial\mu_2}$. As regards the second property, it will be possessed only by the function $\partial\Omega/\partial\lambda$. On the other hand, the functions $\partial\Omega/\partial\mu_1$ and $\partial\Omega/\partial\mu_2$ will be themselves of order $k^2 m_j$, since the first and second terms of the function $\tilde{\Omega}$ are independent of μ_1 and μ_2.

Starting from these remarks and taking (2.83) and (2.84) into account, we obtain the following:

1) Periodic perturbations will produce in our elements (2.32) only those terms of $\tilde{\Omega}$, which contain e_j as a factor. The period of these terms in t is equal to $2\pi/n_j$. On the other hand, terms independent of e_j give only the secular perturbations.

2) The generalized coordinates x_1, x_2, and x_3 will have perturbations of order k^2m_j. As regards y_1, y_2, and y_3, the periodic perturbations in these elements will have $k^2m_je_j$ as a factor.

9. On the Fulfillment of the Orthointerpolational Conditions in the Limited Three-dimensional Elliptical Three-point Problem

To conclude the present chapter we shall dwell on the extent to which one can expect a small planet, which is well within the definition of a bounded three-dimensional elliptical three-point problem, will fulfill satisfactorily the orthointerpolational conditions.

N. D. Moiseyev [13] considered this question as applied to the limited three-dimensional secular three-point problem. He used canonical variables different from ours, and showed that the existence of a Jacobi integral in this equation implies, with certain approximation, the existence of the orthointerpolational conditions. We shall report briefly Moiseyev's results in terms of our canonical vari-

ables (2.32).

In the limited three-dimensional circular three-point problem we assume that e_j, the eccentricity of the orbit of the perturbing point, is equal to zero, and therefore our canonical variables (2.32) will, on the one hand, by virtue of the smallness of the perturbing mass m_j, vary in the corresponding time interval t in nearly the same fashion as these variables vary in the unperturbed motion. On the other hand, these variations will be directly proportional to the time in the case when ortho-interpolational schemes are employed. Both assumptions necessitate a sufficient justification.

First, the differential equations for the general non-averaged bounded three-point problem are written in our variables in the form

$$\left.\begin{aligned} \frac{dx_1}{dt} &= k^2 m_j \frac{\partial W_j}{\partial y_1}, & \frac{dy_1}{dt} &= \frac{k^4}{x_1^3} - n_j - k^2 m_j \frac{\partial W_j}{\partial x_1}, \\ \frac{dx_2}{dt} &= k^2 m_j \frac{\partial W_j}{\partial y_2}, & \frac{dy_2}{dt} &= n_j - k^2 m_j \frac{\partial W_j}{\partial x_2}, \\ \frac{dx_3}{dt} &= k^2 m_j \frac{\partial W_j}{\partial y_3}, & \frac{dy_3}{dt} &= n_j - k^2 m_j \frac{\partial W_j}{\partial x_3}. \end{aligned}\right\} \qquad (2.100)$$

From this we see that the right halves of the equations of (2.100) contain partial derivatives of the function W_j, multiplied by the quantity $k^2 m_j$. If the perturbing body is assumed to be Jupiter, then the last quantity is quite small compared with n_j and k^4/x_1^3. At

AS

50) the same time, the partial derivatives of W_j with respect to y_1, y_2, and y_3 have, as we have already seen, an order of magnitude $W_j = 1/\Delta - (r \cos \vartheta)/r_j^2$ less than unity even under the most favorable case for Ceres, if we use as astronomic units the mass of the sun (equal to unity) and the average solar day. As regards the partial derivatives of W_j with respect to x_1, x_2, and x_3 their order of magnitude is W_j/k. Therefore the right halves of the system (2.100) differ from the right halves of the equations of unperturbed motion:

$$\left.\begin{array}{lll} \frac{dx_1}{dt} = 0, & \frac{dx_2}{dt} = 0, & \frac{dx_3}{dt} = 0, \\ \frac{dy_1}{dt} = \frac{k^4}{x_1^3} - n_j, & \frac{dy_2}{dt} = n_j, & \frac{dy_3}{dt} = n_j, \end{array}\right\} \qquad (2.101)$$ 101

by an amount on the order of km_j. This last quantity is less than 1:50 000 in the astronomic system of units for the case of Jupiter. The quantities k^4/x_1^3 and n_j have in the case of Ceres and Jupiter values 0.00374 and 0.00145. Therefore the solutions of the system (2.100) will also deviate from the solutions of the system (2.101) during a certain time interval by an insignificant amount, if the initial data will be the same for the system (2.100) as for (2.101).

Second, in the analysis of the time dependence of W_j in Section 5 we noted that the totality of the members of

the expansion of W_j, dependent on the time, has e_j as a factor. In our case $e_j = 0$ and therefore W_j does not depend explicitly on the time. Therefore neither the function Ω nor the function $\widetilde{\Omega}$ depend explicitly on the time. In this case the solutions (2.83) and (2.84) for the circular bounded three-point problem have in the orthe-interpolational case the form

$$x_1 = x_{1n} + \frac{\partial\widetilde{\Omega}}{\partial\mu_1}\cdot(t - t_0),\ x_2 = x_{2n} + \frac{\partial\widetilde{\Omega}}{\partial\mu_2}\cdot(t - t_0),\ x_3 = x_{3n} + + \left[m_3^{(1)}\frac{\partial\widetilde{\Omega}}{\partial\mu_1} + m_3^{(2)}\frac{\partial\widetilde{\Omega}}{\partial\mu_2}\right](t - t_0), \tag{2.102}$$

and

$$y_1 = y_{1n} - \frac{\partial\widetilde{\Omega}}{\partial\lambda}\cdot(t - t_0),\ y_2 = y_{2n} - l_2\frac{\partial\widetilde{\Omega}}{\partial\lambda}\cdot(t - t_0),\ y_3 = y_{3n} - l_3\frac{\partial\widetilde{\Omega}}{\partial\lambda}(t - t_0)\cdot \tag{2.103}$$

It is seen from (2.102) and (2.103) that the changes in our canonical varibbles (2.32) will indeed be directly proportional to the time.

We shall show now that the orthointerpolational conditions (2.81) will be satisfied also in the case of a three-dimensional bounded three-point circular problem accurate to second-order quantities with respect to m_j and the squares of the changes of x_i $(i = 1, 2, 3)$ inclusive. Indeed, in the case of a circular bounded three-point problem the equations of motion (2.101) have one first

Jacobi integral

$$\Omega = \frac{k^4}{2x_1^2} + n_j(x_1 - x_2 - x_3) + k^2 m_j W_j = \text{const.} \tag{2.104}$$

As we have already seen, the changes in the quantities x_1, x_2, and x_3 will be of order k^2m_j. Taking the first variation $\delta\Omega$ of the function Ω with respect to x_1, x_2, and x_3 and discarding the small terms of order $k^2m_j\,\delta x_i$ (i = 1, 2, 3) and higher, we obtain

$$\delta\Omega = -\frac{k^4}{x_1^3}\delta x_1 + n_j(\delta x_1 - \delta x_2 - \delta x_3) = 0, \tag{2.105}$$

or

$$-\delta\Omega = \left(\frac{k^4}{x_1^3} - n_j\right)\delta x_1 + n_j\delta x_2 + n_j\delta x_3 = 0. \tag{2.106}$$

Determining l_2 and l_3 from this, we get

$$l_2 = \frac{n_j}{n - n_j}, \quad l_3 = \frac{n_j}{n - n_j}, \tag{2.107}$$

since the coefficient of δx_1 is unity, and

$$n = \frac{k^4}{x_1^3}. \tag{2.108}$$

On the other hand, in view of the smallness of km_j, we obtain from the system (2.100)

$$m_3^{(1)} = -\frac{n - n_j}{n_j}, \quad m_3^{(2)} = -1, \tag{2.109}$$

where n is determined from (2.108). Formulas (2.109) are

48

obviously obtained if we discard in the right halves of (2.100) the terms of order m_j, which, as we have already seen, are several orders of magnitude smaller than the terms k^4/x_1^3 and n_j. Therefore $m_3^{(1)}$ and $m_3^{(2)}$ will be determined essentially by these last terms, i.e., they will have the same values as in the unperturbed motion.

Substituting (2.107) and (2.109) into the orthointerpolational conditions (2.81), we see that this condition will be satisfied accurate to first order in m_j inclusive, since the changes of x_1, x_2, and x_3 will be of the order of m_j.

In the case of the three-dimensional limited elliptical three-point problem, the quantities $m_3^{(1)}$ and $m_3^{(2)}$ will be determined by formulas (2.109) with the same stipulations as made above. On the other hand, for the quantities ℓ_2 and ℓ_3 we have the same formulas (2.07) as for the case of the circular limited three-point problem, as will be clear from the following facts.

For the three-dimensional limited elliptical three-point problem we obtain in place of the Jacobi integral (2.104) the following quasi-integral for the general equations of motion:

$$\Omega - \int_{t=t_0}^{t} \frac{\partial \Omega}{\partial t} \cdot dt = \text{constant}, \qquad (2.110)$$

or, taking the arguments of Section 5 into account

$$\frac{k^4}{2x_1^2} + n_j(x_1 - x_2 - x_3) + k^2 m_j \left[W_j - \int\limits_{M_j = M_{j0}}^{M_j} \frac{\partial W_j}{\partial M_j} dM_j \right] = \text{constant.} \quad (2.111)$$

By taking variations of the left half of (2.111) with respect to x_1, x_2, and x_3, and by discarding the variation of the third term of the left half of (2.111) because of the smallness of m_j, we obtain

$$\delta x_1 + l_2 \delta x_2 + l_3 \delta x_3 = 0, \quad (2.112)$$

where

$$l_2 = \frac{n_j}{n - n_j}, \quad l_3 = \frac{n_j}{n - n_j}, \quad (2.113)$$

with

$$n = \frac{k^4}{x_1^3}. \quad (2.114)$$

Thus, in either the elliptical or in the circular case l_2 and l_3, and also $m_3^{(1)}$ and $m_3^{(2)}$ are determined by the same formulas. From this we see that the orthointerpolational conditions (2.81) will be satisfied in the case of the three-dimensional elliptical three-point problem with the same order of accuracy as in the circular case. Consequently, from the theoretical point of view, the use of the orthointerpolational variants can be fully justified here. It will become clear in the following chapters that it is justified also from the practical

point of view.

CHAPTER III. REDUCTION OF THE OBSERVED DATA FOR PURPOSES OF OBTAINING THE INTERPOLATION ELEMENTS OF MOTION OF CERES λ, μ_1, AND μ_2 IN THE TIME INTERVAL FROM 1801 TO 1938

1. General Remarks on the Preliminary Kepler Elements as the Intermediate Stage in the Determination of the Interpolation Elements from Knowledge of the Totality of the Normal Places

For a practical construction of the interpolational analytical theory of the motion of Ceres, which we described above, we must first obtain from the empirical material the interpolation elements λ, μ_1, and μ_2; we must then verify whether the orthointerpolational conditions are satisfied. To determine the interpolational elements λ, μ_1, and μ_2 as functions of the canonical elements x_1, x_2, x_3; y_1, y_2, y_3 we must determine from the statistically-processed observational data the coefficients ℓ_2, ℓ_3, $m_3^{(1)}$, and $m_3^{(2)}$, the preliminary values of the interpolational elements $\tilde{\lambda}$, $\tilde{\mu}_1$, and $\tilde{\mu}_2$, and the initial values of the osculating elements x_{1n}, x_{2n}, x_{3n}; y_{1n}, y_{2n}, y_{3n}. From the constants ℓ_2, ℓ_3, $m_3^{(1)}$, and $m_3^{(2)}$

we can verify that the orthointerpolational conditions are satisfied [see the preceding chapter, formula (2.81)]. In the present chapter we consider only the question of the determination of the constants:

$$l_2,\ l_3,\ m_3^{(1)},\ m_3^{(2)}, \tag{3.1}$$

and of the quantities

$$\tilde{\lambda},\ \tilde{\mu}_1,\ \tilde{\mu}_2. \tag{3.2}$$

The main difficulty in the determination of the interpolational elements is the fact that the bulk of the observational material are the normal places of Ceres, and must be first converted into the osculating elements of Ceres. However, the determination of six unknown osculating elements can be replaced first by a determination of the Kepler elements of an unperturbed orbit such as would represent exactly three observations separated by a certain time interval. This thought brings to mind the main idea of the method which we have chosen to determine the constants (3.1) and (3.2).

This method of determining the constants (3.1) and (3.2) consists in determining these constants directly from observation, bypassing the determination of the osculating elements. This can be done in the following

fashion.

Assume that we know from observation the spherical geocentric coordinates of a small planet (say the right ascensions α_i and declinations δ_i), for a series of successive time intervals t_i $(i = 1, 2, \ldots, n)$:

$$\left.\begin{array}{lll} t_1 & \alpha_1 & \delta_1, \\ t_2 & \alpha_2 & \delta_2, \\ \cdot & \cdot & \cdot \\ t_n & \alpha_n & \delta_n. \end{array}\right\} \qquad (3.3)$$

Now using the fact that the position of Ceres, i.e., its spherical coordinates α_i and δ_i, can be approximated over the length of all the observations (140 years) accurate to half a degree of arc (this can be seen from what follows) with the aid of the unperturbed Kepler orbit with elements

$$\{\ni_1, \ni_2, \ni_3, \ni_4, \ni_5, \ni_6\} = \{\ni_i\}. \qquad (3.4)$$

For this purpose we can calculate the geocentric positions of the small planet α_{ib} and δ_{ib} for the same instants t_i $(i = 1, 2, \ldots, n)$ with account of the aberration time and using the elements (3.4):

$$\left.\begin{array}{lll} t_1 & \alpha_{1b} & \delta_{1b}, \\ t_2 & \alpha_{2b} & \delta_{2b}, \\ \cdot & \cdot & \cdot \\ t_n & \alpha_{nb} & \delta_{nb}. \end{array}\right\} \qquad (3.5)$$

The quantities α_i and α_{ib} as well as δ_i and δ_{ib} will, generally speaking, differ from each other, but in

most cases by not more than half a degree. We shall assume that the reason for these differences arise only in the fact that we have replaced the osculating values of the elements by their constant unperturbed values in calculating the quantities in table (3.5). Thus, the inaccuracy in the knowledge of the geocentric coordinates of the sun, as well as other factors, will be disregarded.

We now formulate the following procedure.

We consider the calculated and observed values of the coordinates of the small planet for the first three instants of observation.

We have: for t_k: the values of α_k and δ_k are observed while $\alpha_{kb} = \alpha(t_k, \vartheta_i)$ and $\delta_{kb} = (t_k, \vartheta_i)$ are calculated.

$$k = 1, 2, 3. \tag{3.6}$$

We now determine the corrections $\{\Delta \vartheta_i^2\}$ to the elements $\{\vartheta_i\}$, such as to make the quantities $\alpha_{kb}^{(1)}$ and $\delta_{kb}^{(1)}$ (k = 1, 2, 3), calculated with the elements $\{\vartheta_i + \Delta\vartheta_i^2\}$, coincide exactly with α_k and δ_k (k = 1, 2, 3), respectively. These carrections $\{\Delta\vartheta_i^2\}$ can be obtained obviously from the following equations (we are leaving aside for the time being the question of whether these equations can be solved):

$$\left.\begin{aligned}\alpha_k &= \alpha(t_k;\ \ni_1 + \Delta\ni_1^2,\ \ni_2 + \Delta\ni_2^2, \ldots, \ni_6 + \Delta\ni_6^2),\\ \delta_k &= \delta(t_k;\ \ni_1 + \Delta\ni_1^2,\ \ni_2 + \Delta\ni_2^2, \ldots, \ni_6 + \Delta\ni_6^2),\\ k &= 1, 2, 3.\end{aligned}\right\} \tag{3.7}$$

The system of elements $\{\ni_i + \Delta\ni_i^2\}$ can be naturally referred to the epch $t_2' = (t_1+t_2+t_3)/3$.

We shall call this system of elements $\{\ni_i + \Delta\ni_i^2\}$ the preliminary system of elements for the epoch t_2'.

In similar fashion, by combining the second, third, and fourth observation, the third, fourth, and fifth observation, etc., we can find the preliminary systems of elements $\{\ni_i + \Delta\ni_i^3\}$ and $\{\ni_i + \Delta\ni_i^4\}$ ... referred to the epochs $t_3' = (t_2+t_3+t_4)/3$, $t_4' = (t_3+t_4+t_5)/3$ ...

If we have n observations, we can obtain in this manner (n-2) systems of preliminary elements

$$\{\ni_i + \Delta\ni_i^k\},\ k = 2, 3, \ldots, n-1. \tag{3.8}$$

The values obtained for the preliminary elements will be closer to the values of the osculating elements than the values of elements (3, 4). They will be all the greater, the smaller the time interval between three successive observations. However, as shown by experience in computation, a decrease in the time interval is accompanied by a decrease in the accuracy of determination of

the preliminary elements, since the observed α_i and δ_i are subject to a certain error. In practice therefore, the determination of the preliminary elements is satisfactory only when the observations are separated by considerable time interval. We shoose as such a time interval the lapse between two successive oppositions of Ceres.

In the next section we shall consider one of the methods of determining the preliminary elements. We shall show in Section 3 that the preliminary elements determined by this method can be used in first approximation to determine the quantities (3.1) in place of the system of osculating elements, and also to determine the preliminary values of (3.2).

2. Concerning One Method of Determining the First Approximation of the Preliminary Kepler Elements of a Small Planet From Its Normal Places

We have seen in the preceding section that α_b and δ_b are the calculated equatorial goecentric coordinates -- are functions of the time t and of the assumed values of the elements $\{\vartheta_i + \Delta\vartheta_i\}$, where $\{\vartheta_i\}$ denotes the system of elements (3.4):

$$\left.\begin{aligned}\alpha_b &= \alpha(t;\ \ni_1+\Delta\ni_1,\ \ni_2+\Delta\ni_2,\ldots\ldots,\ \ni_6+\Delta\ni_6),\\ \delta_b &= \delta(t;\ \ni_1+\Delta\ni_1,\ \ni_2+\Delta\ni_2,\ldots\ldots,\ \ni_6+\Delta\ni_6).\end{aligned}\right\} \tag{3.9}$$

Expanding the right halves of (3.9) in powers of $\Delta\ni_i$ (i = 1, 2, ..., 6) and confining ourselves to the first powers of $\Delta\ni_i$ (i = 1, 2, ..., 6) we obtain

$$\left.\begin{aligned}\alpha_b &= \alpha(t;\ \ni_i,\ldots\ldots,\ni_6)+\sum_{i=1}^{6}\left.\frac{\partial\alpha}{\partial\ni_i}\right|_{\Delta\ni_i=0}\cdot\Delta\ni_i,\\ \delta_b &= \delta(t;\ \ni_i,\ldots\ldots,\ni_6)+\sum_{i=1}^{6}\left.\frac{\partial\delta}{\partial\ni_i}\right|_{\Delta\ni_i=0}\cdot\Delta\ni_i.\end{aligned}\right\} \tag{3.10}$$

These expressions can be used to determine the preliminary elements. In fact, the k-th system of preliminary elements $\{\ni_i+\Delta\ni_i\}$ is determined from the following equations:

$$\left.\begin{aligned}\alpha_{k+l} &= \alpha(t_{k+l};\ \ni_1+\Delta\ni_1^k,\ldots\ldots,\ni_6+\Delta\ni_6^k\\ \delta_{k+l} &= \delta(t_{k+l};\ \ni_1+\Delta\ni_1^k,\ldots\ldots\ni_6+\Delta\ni_6^k\end{aligned}\right\} \tag{3.11}$$

$$(l=-1,0+1).$$

Using (3.10) we obtain

$$\left.\begin{aligned}\Delta\alpha_{k+l}+\mathfrak{A}_{1,k+l}\cdot\Delta'\ni_1^k+\ldots\ldots+\mathfrak{A}_{6,k+l}\cdot\Delta'\ni_6^k=0,\\ \Delta\delta_{k+l}+\mathfrak{B}_{1,k+l}\cdot\Delta'\ni_1^k+\ldots\ldots+\mathfrak{B}_{6,k+l}\cdot\Delta'\ni_6^k=0,\end{aligned}\right\} \tag{3.12}$$

$$(l=-1,0,+1),$$

where

$$\Delta\alpha_{k+l} = \alpha_{k+l,b} - \alpha_{k+l}, \quad \Delta\delta_{k+l} = \delta_{k+l,b} - \delta_{k+l}, \tag{3.13}$$

(our values of α_{k+l} and δ_{k+l} are given in Table (3.5)),

$$\mathfrak{A}_{m,k+l} = \left.\frac{\partial\alpha}{\partial\vartheta_m}\right|_{\substack{t=t_{k+l} \\ \Delta\vartheta_l = 0}}, \quad \mathfrak{B}_{m,k+l} = \left.\frac{\partial\delta}{\partial\vartheta_m}\right|_{\substack{t=t_{k+l} \\ \Delta\vartheta_l = 0,}}, \tag{3.14}$$

$$m = 1, 2, \ldots, 6; \ l = -1, 0, +1.$$

Expressions (3.12) are none other than a system of six linear algebraic equations with six unknowns $\Delta'\vartheta_i^k$ (i = 1, 2, ..., 6). The coefficients $\mathfrak{A}_{m,k+l}$ and $\mathfrak{B}_{m,k+l}$ are constants in this case. We shall consider the method of determining these coefficients in Section 4.

The determination of the corrections $\Delta'\vartheta_i^k$ themselves (i = 1, 2, ..., 6) from equations (3.12) can be made, for example, by the Gauss method, i.e., by successive elimination of the unknowns.

The procedure described for obtaining the corrections $\Delta'\vartheta_i^k$ (i = 1, 2, ..., 6) will be called here the determination of the preliminary elements in the first approximation. Naturally, this procedure cannot, generally speaking, give exact solutions for equations (3.11), since we have disregarded in the setting up of (3.12) the terms

of order higher than the first in the quantities
(i = 1, 2, ..., 6).

However, for statistical purposes (and this is precisely the type of problem involved in determining (3.1) and (3.2)) this procedure is quite adequate, since the corrections are small and the disregarded quantities will be sufficiently small. In Section 4 we shall estimate the magnitude of the disregarded quantities.

To obtain the exact values of the preliminary elements we can proceed, after obtaining the preliminary elements from the first approximation, in the same fashion as in the method of successive approximations. Namely, we use the values of the preliminary elements obtained in the first approximation, $\{\ni_i + \Delta' \ni_i^k\}$, to calculate again the values of $\alpha^1_{k+\ell,b}$, $\delta^1_{k+\ell,b}$, and $\mathfrak{R}'_{m,k+\ell}$, $\mathfrak{L}'_{m,k+\ell}$ for the instants t_{k-1}, t_k, and t_{k+1}, the values of $\Delta'\alpha_{k+\ell}$ and $\Delta'\delta_{k+\ell}$ are calculated, substituted in (3.12) to determine the new corrections $\Delta''\ni_i^k$, etc. Thus, for each Kepler element $\ni_i$ we obtain by successive approximation the corrections $\Delta'\ni_i^k$, $\Delta''\ni_i^k$, $\Delta'''\ni_i^k$, ... and then the exact values of the preliminary elements $\bar{\ni}_i^k$ are determined from the formula

$$\bar{\ni}_i^k = \ni_i^k + \Delta'\ni_i^k + \Delta''\ni_i^k + \ldots\ldots, \qquad (3.15)$$
$$i = 1, 2, \ldots, 6.$$

Of course, it is still necessary to prove the convergence of this series to the solutions of (3.11).

In our work we confined ourselves only to a determination and a certain estimate of the preliminary elements of Ceres in the first approximation.

3. Use of the Method of Correlation of Many Variables to Determine the Preliminary Values of the Coefficients of the Interpolation Elements and the Most Probable Values of These Interpolation Elements

In the present section we develop a method for calculating the coefficients of the interpolation elements and the most probable values of the interpolation elements themselves, which we employed to construct an interpolational analytic theory for the motion of Ceres. This method is essentially a method of linear correlation of three variables, since, as we have seen in the preceding section, the interpolation elements contain not more than three canonical variables. This method differs from the correlation method for three variables in that the conditional equations contain a certain additive random term (the definition of randomness of this term will be made more pre-

10)

cise later on).

Thus, we can now start from the fact that we have already determined a certain set of preliminary Kepler elements $\{\ni_i + \Delta'\ni_i^k\}$ (i = 1, ..., 6; k = 2, 3, ..., n-1). The preliminary Kepler elements $\ni_i + 2\Delta'\ni_i^k$ are determined, as we have seen, from observations with epochs t_{k-1}, t_k, t_{k+1}. The values of the osculating Kepler elements for these epochs will be denoted by $\{\ni_i + \Delta'\ni_i^k + \overline{\Delta\ni}^{(k,\ell)}\}$ and when $\ell = -1$ we obtain the osculating elements for the epoch t_{k-1}, when $\ell = 0$ we get the values for t_k, and when $\ell = +1$ we get the values for t_{k+1}.

It is obvious that

$$\Delta'\ni_i^{(k)} + \Delta\overline{\ni}_i^{(k,l)} = \Delta'\ni_i^{(k+1)} + \overline{\Delta\ni}_i^{(k+1,l-1)}.$$

Assume now that the following relation is satisfied for the osculating elements $\ni_i + \Delta'\ni_i^{(k)} + \overline{\Delta\ni}^{(k,\ell)}$ for all values of k (k = 2, 3, ..., n-1) and for all values of ℓ (ℓ = - 1, 0, + 1):

$$\sum_{j=1}^{3}[m_j(\ni_j + \Delta'\ni_j^{(k)} + \Delta\overline{\ni}_j^{(k,l)})] + \mathfrak{M} = 0. \tag{3.16}$$

We denote by $\overline{\ni}_i$ (i = 1, 2, ..., 6) the quantities

$$\bar{\ni}_i = \frac{1}{3(n-2)} \sum_{k=2}^{i-1} \sum_{l=-1}^{+1} (\ni_i + \Delta'\ni_i^{(k)} + \overline{\Delta\ni}_i^{(k,l)}), \qquad (3.17)$$

$$i = 1, 2, \ldots, 6.$$

$\bar{\ni}_i$ is obviously some sort of average of n-2 values of the osculating elements $\ni_i + \Delta'\ni_i^{(k)} + \overline{\Delta\ni}_i^{(k,l)}$. It is easy to see that the quantities $\bar{\ni}_i$ satisfy the relations (3.16), i.e.,

$$m_1\bar{\ni} + m_2\bar{\ni}_2 + m_3\bar{\ni}_3 + \mathfrak{M} = 0. \qquad (3.18)$$

In this case, denoting by $\delta\ni_i^{(k)}$ (k = 1, 2, ..., n) the quantities

$$\delta\ni_i^{(k)} = (\ni_i + \Delta'\ni_i^{(k)} + \overline{\Delta\ni}_i^{(k,0)}) - \bar{\ni}_i, \qquad (3.19)$$

$$k = 2, 3, \ldots, n-1,$$

and

$$\delta\ni_i^{(1)} = (\ni_i + \Delta'\ni_i^{(2)} + \overline{\Delta\ni}_i^{(2,-1)}) - \bar{\ni}_i, \qquad (3.20)$$

$$\delta\ni_i^{(n)} = (\ni_i + \Delta'\ni_i^{(n-1)} + \overline{\Delta\ni}_i^{(n-1,+1)}) - \bar{\ni}_i, \qquad (3.21)$$

we find that the quantities $\delta\ni_i^{(k)}$ (k = 1, 2, ..., n) satisfy the following relations (for all values of k):

$$m_1\delta\ni_i^{(k)} + m_2\delta\ni_2^{(k)} + m_3\delta\ni_3^{(k)} = 0. \qquad (3.22)$$

We denote by $\delta'\ni_i^{(k)}$ (k = 2, 3, ..., n-1) the de-

viations of the preliminary elements $\ni_i + \Delta' \ni_i^{(k)}$ from $\bar{\ni}_i$

$$\delta' \ni_i^k = \ni_i + \Delta' \ni_i^{(k)} - \bar{\ni}_i. \tag{3.23}$$

The quantities $\delta' \ni_i^{(k)}$ are the solutions of the equations for the determination of the preliminary Kepler elements, provided one takes as the initial unperturbed values of the elements the quantities $\{\ni_i\}$. This assumption, naturally, can be proved only in first approximation. Let us prove it.

If we take for the initial unperturbed Kepler elements the quantities $\{\ni_i\}$, then the differences between the observed and calculated values $\overline{\Delta\alpha}_{k-1}$, $\overline{\Delta\delta}_{k-1}$, $\overline{\Delta\alpha}_k$, $\overline{\Delta\delta}_k$, $\overline{\Delta\alpha}_{k+1}$, and $\overline{\Delta\delta}_{k+1}$ can be determined in terms of the quantities $\Delta\alpha_{k-1}$, $\Delta\delta_{k-1}$, $\Delta\alpha_k$, $\Delta\delta_k$, $\Delta\alpha_{k+1}$, and $\Delta\delta_{k+1}$, calculated with elements $\{\ni_i\}$, by using the following formulas:

$$\left.\begin{aligned} &\overline{\Delta\alpha}_{k+l} - \Delta\alpha_{k+l} + \mathfrak{A}_{1,k+l}(\bar{\ni}_1 - \ni_1) + \mathfrak{A}_{2,k+l}(\bar{\ni}_2 - \ni_2) + \cdots + \\ &\qquad + \mathfrak{A}_{6,k+l}(\bar{\ni}_6 - \ni_6) = 0, \\ &\overline{\Delta\delta}_{k+l} - \Delta\delta_{k+l} + \mathfrak{B}_{1,k+l}(\bar{\ni}_1 - \ni_1) + \mathfrak{B}_{2,k+l}(\bar{\ni}_2 - \ni_2) + \cdots + \\ &\qquad + \mathfrak{B}_{6,k+l}(\bar{\ni}_6 - \ni_6) = 0. \\ &\qquad l = -1, 0, +1. \end{aligned}\right\} \tag{3.24}$$

But the quantities $\Delta' \ni_i^{(k)}$ are the solutions of the equations

63

$$\left.\begin{array}{c}\Delta\alpha_{k+l} + \mathfrak{A}_{1,k+l}\cdot\Delta'\ni_1^{(k)} + \mathfrak{A}_{2,k+l}\cdot\Delta'\ni_2^{(k)} + \ldots + \mathfrak{A}_{2,k+l}\cdot\Delta'\ni_6^{(k)} = 0,\\ \Delta\delta_{k+l} + \mathfrak{B}_{1,k+l}\cdot\Delta'\ni_1^{(k)} + \mathfrak{B}_{2,k+l}\cdot\Delta'\ni_2^{(k)} + \ldots + \mathfrak{B}_{6,k+l}\cdot\Delta'\ni_6^{(k)} = 0,\\ l = -1,0,+1.\end{array}\right\} \quad (3.25)$$

Adding the right and left halves of the first equation of (3.24), term by term, to the right and left halves of the first equation of the system (3.25), respectively, and following a similar procedure for the remaining five equations of these systems, we obtain, taking (3.23) into account:

$$\left.\begin{array}{c}\overline{\Delta\alpha}_{k+l} + \mathfrak{A}_{1,k+l}\cdot\delta'\ni_1^{(k)} + \mathfrak{A}_{2,k+l}\cdot\delta'\ni_2^{(k)} + \ldots + \mathfrak{A}_{6,k+l}\cdot\delta'\ni_6^{(k)} = 0,\\ \overline{\Delta\delta}_{k+l} + \mathfrak{B}_{1,k+l}\cdot\delta'\ni_1^{(k)} + \mathfrak{B}_{2,k+l}\cdot\delta'\ni_2^{(k)} + \ldots + \mathfrak{B}_{6,k+l}\cdot\delta'\ni_6^{(k)} = 0,\\ l = -1,0,+1.\end{array}\right\} \quad (3.26)$$

This precisely proves the assumption made earlier.

If we now make the assumption that the preliminary elements $\{\ni_i + \Delta'\ni_i^{(k)}\}$ are the average values of the osculating elements for the epochs t_{k-1}, t_k, and t_{k+1}, then, in accordance with the result of the preceding section (extending these results to include systems of preliminary elements for all values of k) we can find the unknown quantities $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_3$, and $\mathfrak{M}$ from the equations

$$\mathfrak{m}_1\delta'\ni_1^{(k)} + \mathfrak{m}_2\delta'\ni_2^{(k)} + \mathfrak{m}_3\delta'\ni_3^{(k)} = 0, \quad (3.27)$$

$$k = 2,3,\ldots,n-1,$$

64

$$\mathfrak{m}_1\bar{\vartheta}_1 + \mathfrak{m}_2\bar{\vartheta}_2 + \mathfrak{m}_3\bar{\vartheta}_3 + \mathfrak{M} = 0, \tag{3.28}$$

for in this case we know the $\bar{\vartheta}_i$:

$$\bar{\vartheta}_i = \frac{1}{n-2}\sum_{k=2}^{n-1}(\vartheta_i + \Delta'\vartheta_i^{(k)}). \tag{3.29}$$

We can, however, forego this assumption. In this case we obtain in place of (3.27)

$$\mathfrak{m}_1\delta'\vartheta_1^{(k)} + \mathfrak{m}_2\delta'\vartheta_2^{(k)} + \mathfrak{m}_3\delta'\vartheta_3^{(k)} = \rho_k, \tag{3.30}$$

where ρ_k will depend on the deviations of the values of the preliminary elements $\{\vartheta_i + \Delta'\vartheta_i^{(k)}\}$ from the mean values of the osculating elements for the instants k - 1, k, and k + 1. Without loss of generality we can put in (3.30) $\mathfrak{m}_1 = 1$. The cases $\mathfrak{m}_2 = 1$ and $\mathfrak{m}_3 = 1$ are considered similarly.

Thus, we are considering an expression in the form

$$\delta'\vartheta_1^{(k)} + \mathfrak{m}_2\delta'\vartheta_2^{(k)} + \mathfrak{m}_3\delta'\vartheta_3^{(k)} = \rho_k, \tag{3.31}$$

The quantities $\bar{\vartheta}_i$ will naturally not be given in this case by the formula (3.29). But if we take the values (3.29) for these quantities and take for $\delta'\vartheta_i^{(k)}$ the values (3.23), the quantities ρ_k in (3.31) will change.

These changed values of ρ_k will be denoted by $\bar{\rho}_k$.

We then obtain in place of (3.31)

$$\delta'\ni_1^{(k)} + m_2\delta'\ni_2^{(k)} + m_3\delta'\ni_3^{(k)} = \bar{\rho}_k. \qquad (3.32)$$

If we know now the values of $\delta'\ni_1^{(k)}$, $\delta'\ni_2^{(k)}$, $\delta'\ni_3^{(k)}$, and $\bar{\rho}_k$, then the coefficients m_2 and m_3 are chosen such as to minimize the sum

$$S^{(1)} = \sum_{k=2}^{n-1} (\delta'\ni_1^{(k)} + m_2\delta'\ni_2^{(k)} + m_3\delta'\ni_3^{(k)} - \bar{\rho}_k)^2. \qquad (3.33)$$

These values of m_2 and m_3 are determined in non-singular cases from the equations

$$\frac{\partial S^{(1)}}{\partial m_2} = 0, \ \frac{\partial S^{(1)}}{\partial m_3} = 0. \qquad (3.34)$$

If we write these equations in expanded form, we can replace $\delta'\ni_i^{(k)}$, $\delta'\ni_2^{(k)}$, $\delta'\ni_k^{(3)}$, and $\bar{\rho}_k$ by the characteristics of their distributions, i.e., by the rms deviations of these quantities σ_1, σ_2, σ_3, and σ_ρ respectively, and by the coefficients of mutual correlation r_{ik} ($i = 1, 2, 3; \rho$; $k = 1, 2, 3, \rho$; $i \neq k$). These characteristics of the distributions are found from the following formulas

66

$$
\left.\begin{array}{l}
\sigma_i^2 = \frac{1}{n-2}\sum_{k=2}^{n-1}(\delta'\ni_i^{(k)})^2,\ i=1,2,3;\ \sigma_\rho^2 = \frac{1}{n-2}\sum_{k=2}^{n-1}(\bar{\rho}_k)^2; \\
r_{ik} = \frac{\sum_{g=2}^{n-1}\delta'\ni_i^{(g)}\delta'\ni_k^{(g)}}{(n-2)\sigma_i\sigma_k},\ i=1,2,3;\ k=1,2,3;\ i\neq k;\ r_{i\rho}=r_{\rho i}=\frac{\sum_{g=2}^{n-1}\delta'\ni_i^{(g)}\bar{\rho}_g}{(n-2)\sigma_i\sigma_\rho} \\
i=1,2,3.
\end{array}\right\} \quad (3.35)
$$

Then equations (3.53) assume the form

$$
\left.\begin{array}{l}
\sigma_1 r_{12} + \mathfrak{m}_2\sigma_2 + \mathfrak{m}_3\sigma_3 r_{32} = \sigma_\rho \cdot r_{\rho 2}, \\
\sigma_1 r_{13} + \mathfrak{m}_2\sigma_2 r_{23} + \mathfrak{m}_3\sigma_3 = \sigma_\rho \cdot r_{\rho 3}.
\end{array}\right\} \quad (3.36)
$$

In the case when all the $\bar{\rho}_k$ vanish and $\sigma_\rho = 0$, the right halves of these equations become equal to zero. Then all the quantities in these equations, with the exception of $\mathfrak{m}_2$ and $\mathfrak{m}_3$, are determined by observation, while $\mathfrak{m}_2$ and $\mathfrak{m}_3$ can be determined from the formulas

$$
\mathfrak{m}_2 = \frac{\sigma_1}{\sigma_2}\cdot\frac{D_{12}}{D_{11}},\ \mathfrak{m}_3 = \frac{\sigma_1}{\sigma_3}\cdot\frac{D_{31}}{D_{11}}, \quad (3,37)
$$

where D_{ik} are the adjoints of the determinant

$$
D = \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}. \quad (3.38)
$$

We solve the problem similarly in the case when the ρ_k are such that

$$
r_{\rho 2} = r_{\rho 3} = 0. \quad (3.39)
$$

67

The $\bar{\delta}_k$ will then be random in the sense of (3.39). Since $\bar{\delta}_k$ are not known directly from observation, we assume the conditions (3.39) to be satisfied when solving this problem.

We can consider the question of determining the rms errors of the coefficients m_2 and m_3. In this case the equations

$$\delta'\vartheta_1^{(k)} + m_2\cdot\delta'\vartheta_2^{(k)} + m_3\delta'\vartheta_3^{(k)} = 0, \tag{3.40}$$

can be regarded as the equation for the regression of the quantities $\delta'\vartheta_2^{(k)}$ and $\delta'\vartheta_3^{(k)}$, with regression coefficients m_2 and m_3 respectively, on the quantity $-\delta'\vartheta_1^{(k)}$. We can therefore use (3.40) to calculate $\delta'\vartheta_1^{(k)}$ from $\delta'\vartheta_2^{(k)}$ and $\delta'\vartheta_3^{(k)}$. The calculated the values of $\delta'\vartheta_1^{(k)}$ will, generally speaking, differ from those obtained by observation. The deviations of the corresponding values from the given ones will be characterized by an rms deviation Σ. From the theory of linear correlation for many variables it is known that this deviation can be determined as follows:

$$\Sigma = \sigma_1\sqrt{\frac{D}{D_{11}}}, \tag{3.41}$$

where D_{11} is the adjoint of the element of the determinant D in the first row and in the first column.

108

It is then sufficient to multiply this rms deviation by $\sqrt{(n-2)/(n-4)}$ to obtain the rms error per unit weight in (3.40). Having found the rms error per unit weight in (3.40), we can easily calculate the rms errors of the coefficients $\mathfrak{m}_2$ and $\mathfrak{m}_3$. For this purpose we must also know the weights of our unknowns. If we convert the formulas for the weights of our unknowns, derived in the theory of normal equations, introducing into these formulas the characteristics of the distribution of the quantities $\delta, \vartheta_i^{(k)}$, then we obtain

$$p_{\mathfrak{m}_2} = (n-2)\cdot\sigma_2^2\cdot D_{11}, \quad p_{\mathfrak{m}_3} = (n-2)\cdot\sigma_3^2\cdot D_{11}. \tag{3.42}$$

The rms errors of the coefficients $\mathfrak{m}_2$ and $\mathfrak{m}_3$ then become

$$\sigma_{\mathfrak{m}_2} = \sqrt{\frac{n-2}{n-4}}\cdot\frac{\Sigma}{\sqrt{p_{\mathfrak{m}_2}}}, \quad \sigma_{\mathfrak{m}_3} = \sqrt{\frac{n-2}{n-4}}\cdot\frac{\Sigma}{\sqrt{p_{\mathfrak{m}_3}}}. \tag{3.43}$$

This method of finding the coefficients $\mathfrak{m}_2$ and $\mathfrak{m}_3$ and their rms errors is quite similar to that described by us in [19].

4. Finite-difference Method of Determining the Derivatives of the Geocentric Equatorial Coordinates from the Osculating Kepler Elements

We now describe the method which we used to determine the quantities $\mathfrak{A}_{ik}$ and $\mathfrak{B}_{ik}$, which enter into the equations for the preliminary elements in the first approximation. These coefficients are determined in the following fashion:

$$\mathfrak{A}_{ik} = \frac{\partial \alpha}{\partial \vartheta_i}\Bigg|_{\substack{t=t_k \\ \Delta\vartheta_i=0,}} , \quad \mathfrak{B}_{ik} = \frac{\partial \delta}{\partial \vartheta_i}\Bigg|_{\substack{t=t_k, \\ \Delta\vartheta_i=0,}} . \qquad (3.44)$$

To calculate these quantities one derives in texts on celestial mechanics, for example in the text by M. F. Subbotin [21], analytic expressions for these quantities in terms of the elements $\{\vartheta_i\}$. However, we can calculate these quantities also without making use of the analytic expressions, using the following procedure.

Assume, for the sake of being definite, that we need to calculate the derivative of α with respect to ϑ_1. For this purpose we calculate three ephemerides with three systems of unperturbed Kepler elements

$$\vartheta_1 - \Delta\vartheta_1,\ \vartheta_2,\ \vartheta_3,\ \vartheta_4,\ \vartheta_5,\ \vartheta_6, \qquad (3.45)$$

$$\vartheta_1 \qquad\quad ,\ \vartheta_2,\ \vartheta_3,\ \vartheta_4,\ \vartheta_5,\ \vartheta_6, \qquad (3.46)$$

$$\vartheta_1 + \Delta\vartheta_1,\ \vartheta_2,\ \vartheta_3,\ \vartheta_4,\ \vartheta_5,\ \vartheta_6. \qquad (3.47)$$

We then obtain the following table (ephemerides) for α:

Моменты времени [1]	Значения по элементам (3.45) [2]	Значения по элементам (3.46) [2]	Значения по элементам (3.47) [2]
t_1	$\alpha_{1b}^{(\text{л})}$	α_{1b}	$\alpha_{1b}^{(n)}$
t_2	$\alpha_{2b}^{(\text{л})}$	α_{2b}	$\alpha_{2b}^{(n)}$
t_k	$\alpha_{kb}^{(\text{л})}$	α_{kb}	$\alpha_{kb}^{(n)}$
t_n	$\alpha_{nb}^{(\text{л})}$	α_{nb}	$\alpha_{nb}^{(n)}$

(3.48)

1) Instants of time; 2) Values from elements.

From Table (3.48) we can set up the following differences for each k (k = 1, 2, ..., n):

$$\begin{array}{ll} \alpha_{kb}^{(\text{л})} & \Delta^{(\text{л})}\alpha_{kb} \\ \alpha_{kb} & \\ & \Delta^{(n)}\alpha^{kb} \\ \alpha_{kb}^{(n)} & \end{array} \qquad (3.49)$$

$$\begin{aligned} \Delta^{(\text{л})}\alpha_{kb} &= \alpha_{kb} - \alpha_{kb}^{(\text{л})}, \\ \Delta^{(n)}\alpha_{kb} &= \alpha_{kb}^{(n)} - \alpha_{kb}. \end{aligned} \qquad (3.50)$$

If we take (3.49) and (3.50) into account and employ the Stirling interpolation formula, we obtain

$$\mathfrak{A}_{1k} = \left.\frac{\partial \alpha}{\partial Э_1}\right|_{\substack{t=t_k \\ \Delta Э_i = 0}} = \frac{1}{2}\left(\Delta^{(\text{л})}\alpha_{kb} + \Delta^{(n)}\alpha_{kb}\right). \qquad (3.51)$$

We can determine the remaining $\mathfrak{A}_{ik}$ and all the $\mathfrak{L}_{ik}$ similarly.

This method of determining the coefficients $\mathfrak{A}_{ik}$ and $\mathfrak{L}_{ik}$ will be called the finite-difference method of determining the derivatives of α and δ from the osculating

elements. This method together with the method for determining the preliminary elements in first approximation is quite similar in its idea to the method of improving orbits by varying the geocentric distances.

The method just described for the determination of the coefficients $\mathfrak{A}_{ik}$ and $\mathfrak{L}_{ik}$ makes it also possible to estimate the influence of terms of order higher than the first, which we have discarded in the derivation of (3.10) in Section 2.

Actually, a greater or smaller value of the difference $\Delta^{(l)}\alpha_{kb} - \Delta^{(p)}\alpha_{kb}$ will always tell us whether the discarded terms have a greater or lesser influence. Specific estimates of these discarded terms will be given by us in the next section.

5. Results of the Calculations. Numerical Values of the Coefficients of the Interpolation Elements of Ceres and Mean Values of the Preliminary Kepler Elements Obtained from the Aggregate of the Normal Places of Ceres During the Time Interval 1801--1938

The preliminary elements of Ceres are calculated in the following fashion.

1. The calculations are based on the normal places of Ceres, numbered 1--9, 47--52 (Table 1 at the end of

this article), 4, 12, 19, 23, 27, and 30--34, 36, 38, and 39 (Table 2).

For the initial elements $\{\partial_i\}$ we took the average elements of Ceres from the work by V. F. Proskurin [14]:

Epoch 1850, January, 0. 0.

$$\left.\begin{array}{llll} M_0 = 160°59'42''.4 & \varphi = 4°29'56''.9 \\ \Omega = 80\ 48\ 31.7 & n = 770''.724 \\ \omega = 67\ 41\ 23.0 & e = 0.078444 \\ i = 10\ 37\ 08.2 & a = 2.767399 \end{array}\right\}. \qquad (3.52)$$

Elements n and a were rounded off for convenience in calculation. In the work by V. F. Proskurin [14]

$$n = 770''.723907, \quad a = 2.7673993.$$

The elements Ω, ω, i of the system (3.52) are referred to the ecliptic and to the equinox of the epoch, and were recalculated for us to the ecliptic and equinox of 1950.0. The results of the calculations are:

$$\left.\begin{array}{l} \Omega_{1950.0} = 82°\ 8'\ 6''.2 \\ \omega_{1950.0} = 67\ 45\ 38.5 \\ i_{1950.0} = 10\ 37\ 10.2 \end{array}\right\}. \qquad (3.53)$$

2. We earmarked for correction the following elements: φ, n, Ω, ω, M_0, and i. The derivatives of α and δ with respect to these elements were calculated by the method described in the preceding section. For this

purpose the elements φ, Ω, ω, M_0, and i were given variations of 1000.0" in the positive and negative directions, and the element n was varied by 0.1". Therefore, in addition to the main ephemerides with elements (3.52) and (3.53), ephemerides with twelve other systems of elements were calculated for the epochs of the normal places.

3. For all thirteen systems of the elements we calculated the quantities P_x, P_y, P_z, Q_x, Q_y, and Q_z in accordance with the formulas

$$\left.\begin{aligned} P_x &= A_1 \cos\omega + A_2 \sin\omega, & Q_x &= A_2 \cos\omega - A_1 \sin\omega, \\ P_y &= B_1 \cos\omega + B_2 \sin\omega, & Q_y &= B_2 \cos\omega - B_1 \sin\omega, \\ P_z &= C_1 \cos\omega + C_2 \sin\omega, & Q_z &= C_2 \cos\omega - C_1 \sin\omega, \end{aligned}\right\} \tag{3.54}$$

where

$$\left.\begin{aligned} A_1 &= \cos\Omega, & B_1 &= \sin\Omega\cos\varepsilon \\ A_2 &= -\cos i \sin\Omega, & B_2 &= \cos i \cos\Omega\cos\varepsilon - \sin i \sin\varepsilon, \\ C_1 &= \sin\Omega\sin\varepsilon, \\ C_2 &= \cos i \cos\Omega\sin\varepsilon + \sin i \cos\varepsilon. \end{aligned}\right\} \tag{3.55}$$

$$P_x^2 + P_y^2 + P_z^2 = 1, \quad Q_x^2 + Q_y^2 + Q_z^2 = 1, \quad P_xQ_x + P_yQ_y + P_zQ_z = 0. \tag{3.56}$$

We then found the quantities

$$\left.\begin{matrix} aP_x, & a\sin\varphi\cdot P_x, & a\cos\varphi\cdot Q_x, \\ aP_y, & a\sin\varphi\cdot P_y, & a\cos\varphi\cdot Q_y, \\ aP_z, & a\sin\varphi\cdot P_z, & a\cos\varphi\cdot Q_z. \end{matrix}\right\} \tag{3.57}$$

4. In the calculation of the ephemerides we used the following rule for taking into account the planetary aber-

ration [21]:

"The true direction to the star at the instant t is the direction in which we see from the position of the earth, at the instant t, the position of the star at the instant $t^0 = t - A\rho$, where A is the planetary aberration constant and ρ is the distance from the earth to the star."

Therefore the ephemerides were calculated by the formulas

$$\left.\begin{aligned} M &= M_0 + n(t - t_0), \ E - e\sin E = M, \\ \rho\cos\alpha\cos\delta &= X + aP_x\cos E - aP_x\sin\varphi + aQ_x\cos\varphi\cdot\sin E, \\ \rho\sin\alpha\cos\delta &= Y + aP_y\cos E - aP_y\sin\varphi + aQ_y\cos\varphi\cdot\sin E, \\ \rho\sin\delta &= Z + aP_z\cos E - aP_z\sin\varphi + aQ_z\cos\varphi\cdot\sin E. \end{aligned}\right\} \quad (3.58a)$$

After determining ρ from these formulas, we obtain further

$$\left.\begin{aligned} M &= M_0 + n(t - t_0) - nA\rho, \ E - e\sin e = M, \\ \rho\cos\alpha\cos\delta &= X + aP_x\cos E - aP_x\sin\varphi + aQ_x\cos\varphi\cdot\sin E, \\ \rho\sin\alpha\cos\delta &= Y + aP_y\cos E - aP_y\sin\varphi + aQ_y\cos\varphi\cdot\sin E, \\ \rho\sin\delta &= Z + aP_z\cos E - aP_z\sin\varphi + aQ_z\cos\varphi\cdot\sin E. \end{aligned}\right\} \quad (3.58b)$$

Formulas (3.58a) serve here to calculate the mutual distance between Ceres and the earth, ρ, which is used in (3.58b) to account for the planetary aberration in M. The geocentric coordinates of the sun X, Y, and Z are taken for the instant t. For the epochs of Fig. 1 they

75

are given in Table 2.

Calculations with the aid of formulas (3.58) were carried out to four decimal places, while those with formulas (3.58) -- with six decimal places. After calculating the main ephemerides with elements (3.52) and (3.53) as well as 12 ephemerides for the calculation of $\mathfrak{a}_{ik}$ and $\mathfrak{b}_{ik}$ in the equations for the preliminary elements, the following were calculated in the first approximation: $\Delta^{(l)}\alpha_{kv}$, $\Delta^{(p)}\alpha_{kv}$ and $\Delta^{(l)}\delta_{kv}$ and $\Delta^{(p)}\delta_{kv}$ for each of the elements φ, n, M_0, Ω, i, and ω.

This was followed by calculation of all the $\mathfrak{a}_{ik}$ and $\mathfrak{b}_{ik}$ by means of formula (3.51) and the similar formulas for the other elements.

6. The equations for the preliminary elements were solved by the Gauss method, i.e., by the method of successive elimination of the unknowns, and the eliminated unknown was expressed in terms of the remaining unknown from the same equation, in which this eliminated unknown was contained with maximum coefficient.

The values of the preliminary element n, φ, $\Delta' M_0$, $\Delta' i$, $\Delta'\Omega$, and $\Delta'\omega$ are listed in Table 3 of page 82 [of source]. The last column of this table indicates the epochs of the preliminary elements, i.e., the quantities $t'_k = (t_{k-1} + t_k + t_{k+1})/3$.

76

The elements Ω, ω, and i were then referred to the plane of the orbit of Jupiter. This was done by means of the formulas

$$\left.\begin{aligned} \mathrm{I} &= \sin\frac{i}{2}\cdot\sin\frac{\Omega+\Delta\omega}{2} = \sin\frac{\Delta\Omega}{2}\sin\frac{i_e+i_j}{2}, \\ \mathrm{II} &= \sin\frac{i}{2}\cdot\cos\frac{\Omega+\Delta\omega}{2} = \cos\frac{\Delta\Omega}{2}\sin\frac{i_e-i_j}{2}, \\ \mathrm{III} &= \cos\frac{i}{2}\cdot\sin\frac{\Omega-\Delta\omega}{2} = \sin\frac{\Delta\Omega}{2}\cos\frac{i_e+i_j}{2}, \\ \mathrm{IV} &= \cos\frac{i}{2}\cdot\cos\frac{\Omega-\Delta\omega}{2} = \cos\frac{\Delta\Omega}{2}\cos\frac{i_e-i_j}{2}. \end{aligned}\right\} \qquad (3.59)$$

The geometrical meaning of the quantities i, Ω, $\Delta\omega$, $\Delta\Omega$, i_e, and i_j is clear from Fig. 2.

Here:

EE' -- the intersection of the plane of the ecliptic with the celestial sphere;

JJ' -- intersection of the plane of the orbit of Jupiter with the celestial sphere;

CC' -- intersection of the celestial sphere with the plane of the orbit of Ceres

$$i = \angle Ц\Omega'J',\ \Omega = \breve{\Omega_j\Omega},\ \Delta\omega = \breve{\Omega_e\Omega},\ \Delta\Omega = \breve{\Omega_j\Omega_e},\ i_e = \angle\Omega_e E'\Omega_e i_j = \angle\Omega\Omega_j\Omega_e,$$

π and π_j are the perihelions of Ceres and Jupiter.

The quantities $\Delta\Omega$, i_e, and i_j are known. Therefore the right halves of (3.59) are also known from which we determine the following quantities: i -- inclination of

77

the plane of the orbit of Ceres to the plane of the orbit of Jupiter, Ω -- longitude of the node of Ceres on the plane of the orbit of Jupiter, measured from the point Ω_j, and $\Delta\omega$ -- angular distance from Ω_e to Ω. The average elements of the orbit of Jupiter were taken by us from Hill [14]:

66)

$$\begin{aligned} &M_{j0} = 148^\circ\ 1'58''.33 && i_j = 1^\circ 18' 41''.81 && e_j = 0.04825382 \\ &\Omega_j = 98\ 55\ 58.16 && \varphi_j = 2\ 45\ 56.93 && a_j = 5.2028029 \\ &\omega_j = 272\ 58\ 28.56 && n_j = 299''.12837656 && m_j = 1 : 1047.355 \end{aligned} \qquad (3.60)$$

Epoch 1850, January 0. 0, GMT

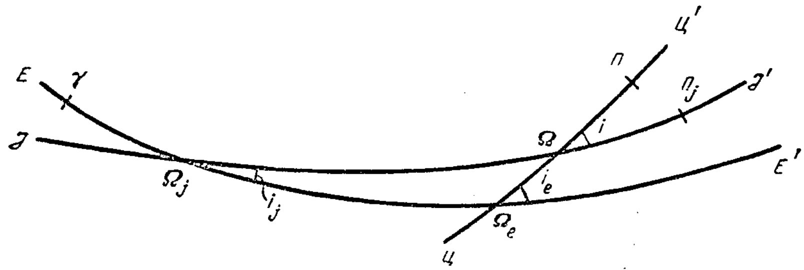

Fig. 2.

The elements Ω_j, ω_j, and i_j are referred here to the ecliptic and to the equinox of the epoch. If we refer them to the 1950.0 epoch, we obtain

$$\Omega_{j1950.0} = 99^\circ 46' 42''.3;\ \omega_{j1950.0} = 273^\circ 31' 30'';\ i_{j1950.0} = 1^\circ 18' 29''.1. \quad (3.61)$$

After obtaining i, Ω, and $\Delta\omega$ we can calculate for each preliminary system the quantities

78

$$
\left.
\begin{aligned}
&M = M_0 + n(t'_k - t_0), \\
&y_1 = M + \omega_e - \Delta\omega + \Omega - l_j, \qquad \frac{x_1}{k} = \sqrt{a}, \\
&y_2 = l_j - \omega_e + \Delta\omega - \Omega, \qquad \frac{x_2}{k} = \sqrt{a} - \sqrt{p} = 2\sqrt{a}\sin^2\frac{\varphi}{2}, \\
&y_3 = l_j - \Omega, \qquad \frac{x_3}{k} = \sqrt{p} - \sqrt{p}\cos i = 2\sqrt{p}\sin^2\frac{i}{2}, \\
&l_1 = M_j + \omega_j, \; M_j = M_{j0} + n_j(t'_k - t_0), \\
&\sqrt{a} = \left(\frac{k}{n}\right)^{1/3}
\end{aligned}
\right\} \quad (3.62)
$$

A table of these quantities is appended to the present article.

In examining the material of this table and the preceding ones, we can note that the corrections for the first, fourth, fifth, sixth, and seventh systems of the preliminary elements are found to be very large, this being evidence in these cases of the large influence of terms of order greater than the first in the equations for the determination of the preliminary Kepler elements. Therefore, in determining the coefficients in the interpolational elements the corresponding systems of the preliminary elements were eliminated. After processing the material of Table 4 by the methods already described in the preceding sections, we find that the coefficients in the interpolational elements (2.7), (2.8), (2.9) (see Chapter II) are

$$
\begin{aligned}
&l_1 = 1, && m_1^{(1)} = 1, \\
&l_2 = +.634\,(\pm.004), && m_3^{(1)} = -1.57549378\,(\pm.000004), \\
&l_3 = +.635\,(\pm.004), && m_2^{(2)} = 1. \\
& && m_3^{(2)} = -\ \ .99830367\,(\pm.000002)
\end{aligned}
\tag{3.63}
$$

(the parentheses contain the rms errors of the individual coefficients). In addition, the following mean values of the preliminary elements were found:

$$
\left.
\begin{aligned}
\frac{x_1}{k} &= 1.663602, & y_1 &= 2533\overset{\circ}{.}525, \\
\frac{x_2}{k} &= \ \ .0052111, & y_2 &= 1522.846, \\
\frac{x_3}{k} &= \ \ .0221027, & y_3 &= 1594.833.
\end{aligned}
\right\}
\tag{3.64}
$$

CHAPTER IV. INTERPOLATION NON-ANALYTICAL THOERY OF MOTION OF CERES IN THE TIME INTERVAL 1801-1938 COMPARISON OF THEORY WITH EXPERIMENT

1. Fulfillment of the Interpolational Condition for the Determined Coefficients of the Interpolational Elements. Certain Changes in the Formulation of the Problem

In Chapter II we have developed a theoretical foundation for the construction of an interpolative-analytical theory for the motion of Ceres. In Chapter III we pre-

sented a method for preliminary determination of the coefficients of the interpolational elements and indeed we determined these coefficients. It is interesting to see whether these determined coefficients satisfy the orthointerpolational conditions (formula (2.81) of Chapter II). We have obtained

$$1 + l_3 m_3^{(1)} = .000 \pm .006, \quad l_2 + l_3 m_3^{(2)} = .001 \pm .008. \tag{4.1}$$

Thus, the interpolational conditions are actually satisfied with great accuracy in the case of the motion of Ceres. However, we must make here one remark, namely that the coefficients l_2 and l_3 have been obtained in the preceding chapter with low accuracy (only two or three true significant figures), whereas, as can be seen from formula (2.84) of Chapter II and was already noted in that chapter, the quantities y_2 and y_3 vary approximately in proportion to the time and furthermore with considerable speed, and consequently the values of l_2 and l_3 must be determined more accurately. Among the possible variants of obtaining more accurate values, we can consider one in which we can modify somewhat the formulation of our problem. Namely, we can assume that the orthointerpolational conditions are satisfied exactly, and determine l_2 and l_3 from the well known values of

21

the quantities $m_3^{(1)}$ and $m_3^{(2)}$. In this case ℓ_2 and ℓ_3 have the following values

$$l_2 = +.63364494. \quad l_3 = +.63472164. \tag{4.2}$$

The values of (4.2) do not contradict, within the limits of accuracy of the computation, the values of ℓ_2 and ℓ_3 from (3.63). This enables us to employ, in constructing an analytic theory of motion of Ceres, the orthointerpolational variant of the limited three-dimensional elliptical three-point problem, developed in the fifth [error in the original; should read "second"] chapter.

2. Calculation of the Laplace Coefficients

In Chapter II we have seen that in the perturbation function W_j, in which we have confined ourselves to terms of second order inclusive relative to e, e_j, and η, there enter the quantities $A^{(i)}$, $A_1^{(i)}$, $A_2^{(i)}$, and $B^{(i)}$, which in final analysis depend on $\tilde{\lambda} = x_{10} + \ell_2 x_{20} + \ell_3 x_{30}$ (a constant quantity) within the limits of averaging over x_2 and x_3. In addition, in order to calculate the perturbations in y_1, y_2, and y_3, it is also necessary to know the derivatives with respect to λ or the quantities $A^{(i)}$, $A_1^{(i)}$, $A_2^{(i)}$, and $B^{(i)}$. In Chapter II we gave the following expressions for these quantities in terms of

the Laplace coefficients:

$$\left.\begin{aligned} a_j A^{(i)} = b_{1,0}^{(i)}, \quad a_j A_2^{(i)} = \frac{1}{1.2}\cdot b_{1,2}^{(i)}, \quad a_j B^{(i)} = \alpha b_{3,0}^{(i)}, \\ a_j A_1^{(i)} = b_{1,1}^{(i)}, \quad a_j A_3^{(i)} = \frac{1}{1.2.3} b_{1,3}^{(i)}, \quad a_j B_1^{(i)} = \alpha\,(b_{3,0}^{(i)} + b_{3,1}^{(i)}). \end{aligned}\right\} \tag{4.3}$$

At first we use the formula

$$\begin{gathered} \frac{1}{2} b_{n,0}^{(i)} = \frac{n(n+2)\dots(n+2i-2)}{2.4.6\dots(2i)}\,\alpha^i (1-\alpha^2)^{-\frac{n}{2}} \cdot \left[1 + \frac{n}{2}\cdot\frac{n-2}{2i+2}\cdot p + \right. \\ \left. + \frac{n(n+2)}{2.4}\cdot\frac{(n-2)(n-4)}{(2i+2)(2i+4)}\cdot p^2 + \dots\right], \\ p = \frac{\alpha^2}{1-\alpha^2}. \end{gathered} \tag{4.4}$$

which is given on page 254 of [15], to calculate, for n=1, the values of $b_{1,0}^{(i)}$ for i = 10, 11, 19, 20, 25, and 26. (In the text [15] the formulas for $b_{1,0}^{(10)}$ and $b_{1,0}^{(11)}$ are given incorrectly). We obtained here

$$\begin{aligned} b_{1,0}^{(10)} &= .000748338, & b_{1,0}^{(19)} &= .1869462.10^{-6}, & b_{1,0}^{(25)} &= .370219.10^{-7}, \\ b_{1,0}^{(11)} &= .000380244, & b_{1,0}^{(20)} &= .969802\ .10^{-6}. & b_{1,0}^{(26)} &= .193173.10^{-7}. \end{aligned} \tag{4.5}$$

(here and throughout the calculations are based on the mean values of the preliminary elements, given in the preceding chaper).

After this, using the recurrence formula

$$b_{1,0}^{(i-1)} = \frac{2i}{2i-1}\cdot(\alpha + \alpha^{-1})\,b_{1,0}^{(i)} - \frac{2i+1}{2i-1}\cdot b_{1,0}^{(i+1)}, \tag{4.6}$$

(we have taken this formula from [15]), we obtain the

values of $b_{1,0}^{(i)}$ for all the remaining values of i, and we went here from the large values of i to the smaller ones, since if the opposite direction is chosen the accuracy is rapidly decreased. (This opposite course is precisely the one recommended in the text of M. F. Subbotin [15].)

We then use the formulas

$$\left.\begin{array}{l} n(1-\alpha)^2\left(b_{n+2,0}^{(i)}+b_{n+2,0}^{(i+1)}\right)=(2i+n)\,b_{n,0}^{(i)}-(2i-n+2)\,b_{n,0}^{(i+1)}, \\ n(1-\alpha)^2\left(b_{n+2,0}^{(i)}-b_{n+2,0}^{(i+1)}\right)=(2i+n)\,b_{n,0}^{(i)}+(2i-n+2)\,b_{n,0}^{(i+1)} \end{array}\right\} \quad (4.7)$$

(formulas (15) on page 257 of [15] are incorrect) to calculate $b_{3,0}^{(i)}$, $b_{5,0}^{(i)}$, and $b_{7,0}^{(i)}$.

We can then find the quantities

$$\left.\begin{array}{ll} b_{1,1}^{(i)}=\frac{\alpha}{2}\left(b_3^{(i-1)}+b_3^{(i+1)}-2\alpha b_3^{(i)}\right), & b_{1,2}^{(i)}=\frac{\alpha}{2}\left[b_{3,1}^{(i-1)}+b_{3,1}^{(i+1)}-2\alpha\left(b_{3,0}^{(i)}+b_{3,1}^{(i)}\right)\right] \\ b_{3,1}^{(i)}=\frac{3\alpha}{2}\left(b_5^{(i-1)}+b_5^{(i+1)}-2\alpha b_5^{(i)}\right), & b_{3,2}^{(i)}=\frac{3\alpha}{2}\left[b_{5,1}^{(i-1)}+b_{5,1}^{(i+1)}-2\alpha\left(b_{5,0}^{(i)}+b_{5,1}^{(i)}\right)\right] \\ b_{5,1}^{(i)}=\frac{5\alpha}{2}\left(b_7^{(i-1)}+b_7^{(i+1)}-2\alpha b_7^{(i)}\right), & b_{1,3}^{(i)}=\frac{\alpha}{2}\left[b_{3,2}^{(i-1)}+b_{3,2}^{(i+1)}-2\alpha\left(b_{3,1}^{(i)}+b_{3,2}^{(i)}\right)\right] \end{array}\right\} \quad (4.8)$$

(These expressions can be obtained by using the formula

$$\frac{db_{n,0}^{(i)}}{d\alpha}=\frac{n}{2}\left(b_{n+2,0}^{(i-1)}+b_{n+2,0}^{(i+1)}-2\alpha\cdot b_{n+2,0}^{(i)}\right),$$

given in [15].)

After finding these quantities, we can find the

values of $A^{(i)}$, $A_1^{(i)}$, $A_2^{(i)}$, $A_3^{(i)}$, and $B^{(i)}$ by means of formulas (4.3) and the following quantities [16]:

$$\left.\begin{aligned}&\frac{\partial A^{(i)}}{\partial a}=\frac{1}{a}\cdot A_1^{(i)}, &&\frac{\partial A_2^{(i)}}{\partial a}=\frac{1}{a}\left(3A_3^{(i)}+2A_2^{(i)}\right),\\ &\frac{\partial A_1^{(i)}}{\partial a}=\frac{1}{a}\left(2A_2^{(i)}+A_1^{(i)}\right), &&\frac{\partial B^{(i)}}{\partial a}=\frac{1}{a_j^2}\left(b_{3,0}^{(i)}+b_{3,1}^{(i)}\right).\end{aligned}\right\}\tag{4.9}$$

Knowing the quantities (4.9) we can find the derivatives of the quantities (4.3) with respect to λ. These have been calculated from the formulas

$$\left.\begin{aligned}&\frac{\partial A^{(i)}}{\partial\lambda}=\frac{\partial A^{(i)}}{\partial x_1}=\frac{\partial A^{(i)}}{\partial a}\cdot\frac{2\sqrt{a}}{k}, &&\frac{\partial A_2^{(i)}}{\partial\lambda}=\frac{\partial A_2^{(i)}}{\partial x_1}=\frac{\partial A_2^{(i)}}{\partial a}\cdot\frac{2\sqrt{a}}{k},\\ &\frac{\partial A_1^{(i)}}{\partial\lambda}=\frac{\partial A_1^{(i)}}{\partial x_1}=\frac{\partial A_1^{(i)}}{\partial a}\cdot\frac{2\sqrt{a}}{k}, &&\frac{\partial B^{(i)}}{\partial\lambda}=\frac{\partial B^{(i)}}{\partial x_1}=\frac{\partial B^{(i)}}{\partial a}\cdot\frac{2\sqrt{a}}{k},\end{aligned}\right\}\tag{4.10}$$

where k is the Gaussian constant.

3. Averaging over the Variables x_1 and x_3 of the Perturbation Function and its Partial Derivative with Respect to .

In Chapter II we have seen that after making the substitution

$$x_1=\lambda-l_2x_2-l_3x_3,\tag{4.11}$$

only the coefficients of the cosines in the perturbation function will depend on x_2 and x_3, i.e., in final analysis

the following quantities

$$A^{(i)},\ A_1^{(i)},\ A_2^{(i)},\ A^{(i)}e,\ A_1^{(i)}e,\ A_2^{(i)}e,\ A^{(i)}e^2,\ A_1^{(i)}e^2,\ A_2^{(i)}e^2,\ B^{(i)}\eta^2,\ \alpha, \alpha e,\ \alpha e^2,\ \alpha\eta^2, \tag{4.12}$$

and similarly

$$\frac{\partial}{\partial\lambda}A^{(i)},\ \frac{\partial}{\partial\lambda}A_1^{(i)},\ \frac{\partial}{\partial\lambda}A_2^{(i)},\ \frac{\partial}{\partial\lambda}(A^{(i)}e),\ \frac{\partial}{\partial\lambda}(A_1^{(i)}e),\ \frac{\partial}{\partial\lambda}(A_2^{(i)}e),\ \frac{\partial}{\partial\lambda}(A^{(i)}e^2),\ \frac{\partial}{\partial\lambda}(A_1^{(i)}e^2),$$
$$\frac{\partial}{\partial\lambda}(A_2^{(i)}e^2),\ \frac{\partial}{\partial\lambda}(B^{(i)}\eta^2),\ \frac{\partial\alpha}{\partial\lambda},\ \frac{\partial}{\partial\lambda}(\alpha e),\ \frac{\partial}{\partial\lambda}(\alpha e^2),\ \frac{\partial}{\partial\lambda}(\alpha\eta^2). \tag{4.13}$$

At first we find the average values of the quantities that do not depend on e or η. Confining ourselves to the first terms we obtain (the averaged quantities are denoted by two superior bars):

$$\left.\begin{aligned}
&\overline{\overline{A}}^{(i)} = A^{(i)} + \frac{\partial A^{(i)}}{\partial \widetilde{x}_2}\cdot\delta\widetilde{x}_2 + \frac{\partial A^{(i)}}{\partial \widetilde{x}_3}\cdot\delta\widetilde{x}_3,\\
&\text{similarly for}\ \overline{\overline{A}}_1^{(i)},\ \overline{\overline{A}}_2^{(i)},\ \overline{\overline{\alpha}},\\
&\frac{\partial\overline{\overline{A}}^{(i)}}{\partial\lambda} = \frac{\partial A^{(i)}}{\partial\lambda} + \frac{\partial^2 A^{(i)}}{\partial\lambda\partial\widetilde{x}_2}\cdot\delta\widetilde{x}_2 + \frac{\partial^2 A^{(i)}}{\partial\lambda\partial\widetilde{x}_3}\cdot\delta\widetilde{x}_3,\\
&\text{similarly for}\ \frac{\partial\overline{\overline{A}}_1^{(i)}}{\partial\lambda},\ \frac{\partial\overline{\overline{A}}_2^{(i)}}{\partial\lambda},\ \frac{\partial}{\partial\lambda}(\overline{\overline{\alpha}}),
\end{aligned}\right\} \tag{4.14}$$

and, as it turns out, the discarded terms do not influence the result. In formulas (4.14) we have

86

$$\left.\begin{aligned}
&\frac{\partial A^{(i)}}{\partial \widetilde{x}_2} = \frac{\partial A^{(i)}}{\partial a}\cdot 2\sqrt{a}\sqrt{a_j}(-l_2), \quad \frac{\partial^2 A^{(i)}}{\partial\lambda\partial\widetilde{x}_2} = \left(\frac{\partial^2 A^{(i)}}{\partial a^2}\cdot 2a + \frac{\partial A^{(i)}}{\partial a}\right)\cdot\frac{2}{k}\sqrt{a_j}(-l_2).\\
&\text{similarly for } A_1^{(i)},\ A_2^{(i)},\\
&\frac{\partial\alpha}{\partial\widetilde{x}_2} = \frac{2\sqrt{a}}{\sqrt{a_j}}(-l_2), \quad \frac{\partial^2\alpha}{\partial\lambda\partial\widetilde{x}_2} = \frac{1}{\sqrt{\alpha}}\cdot(-l_2)\frac{2\sqrt{a}}{ka_j}, \quad \widetilde{x}_2 = \frac{x_2}{k\sqrt{a_j}},\\
&\delta\widetilde{x}_2 = \frac{1}{2k\sqrt{a_j}}\left[(\overline{x}_2 - x_{2cp}) - (x_{2cp} - \underline{x}_2)\right],
\end{aligned}\right\} \tag{4.15}$$

$$\left.\begin{aligned}
&\frac{\partial A^{(i)}}{\partial \widetilde{x}_3} = \frac{\partial A^{(i)}}{\partial a}\cdot 2\sqrt{a}\sqrt{a_j}(-l_3), \quad \frac{\partial^2 A^{(i)}}{\partial\lambda\partial\widetilde{x}_3} = \left(\frac{\partial^2 A^{(i)}}{\partial a^2}\cdot 2a + \frac{\partial A^{(i)}}{\partial a}\right)\cdot\frac{2}{k}\sqrt{a_j}(-l_3).\\
&\text{similarly for } A_1^{(i)},\ A_2^{(i)},\\
&\frac{\partial\alpha}{\partial\widetilde{x}_3} = \frac{2\sqrt{a}}{\sqrt{a_j}}(-l_3), \quad \frac{\partial^2\alpha}{\partial\lambda\partial\widetilde{x}_3} = \frac{1}{\sqrt{\alpha}}(-l_3)\cdot\frac{2\sqrt{\alpha}}{ka_j}, \quad \widetilde{x}_3 = \frac{x_3}{k\sqrt{a_j}}.\\
&\delta\widetilde{x}_3 = \frac{1}{2k\sqrt{a_j}}\left[(\overline{x}_3 - x_{3cp}) - (x_{3cp} - \underline{x}_3)\right],
\end{aligned}\right\}$$

and the quantities x_{2av} and x_{3av} are taken from (3.64) of Chapter III. (4.16)

To determine the upper and lower limits ($\overline{x}_2$ and $\overline{x}_3$ and $\underline{x}_2$ and $\underline{x}_3$, respectively) of the quantities x_2 and x_3 we proceeded as follows.

Assume that we have obtained, by processing the observational data, the values of the preliminary element x_2 (we can reason similarly for x_3). As the average value of x_2 we have taken the average of x_2 from Table 4. We can verify from this table, however, that the distribution of the quantity x_2 about this average value is not symmetrical. Indeed, we can calculate the mean squared deviations

87

$$\sigma_{-}^{2} = \frac{1}{n_5} \sum_{q=n_1}^{n_2} (x_{2q} - x_{2cp})^2, \tag{4.17}$$

$$\sigma_{+}^{2} = \frac{1}{n_6} \sum_{q=n_3}^{n_4} (x_{2q} - x_{2cp})^2, \tag{4.18}$$

where x_{2q} are the values of the element x_2 for the instant t_q $(q = 1, 2, \ldots, n)$. Those values of x_{2q}, which are smaller than x_{2av}, enter into the determination of the quantities σ_{-}, and their number is n_5, while the remaining quantities (whose number is n_6 and which are greater than x_{2av}) enter into the determination of σ_{+}. If the distribution is symmetrical, the quantities σ_{-} and σ_{+} should coincide. However, for the quantities x_2 and x_3 they do not coincide. This can be verified by calculating σ_{-} and σ_{+} for the values of Table 4 or the aforementioned work. We obtain:

$$\text{for } \left| x_2 : \begin{array}{l} \sigma_{-} = .0003190.k, \\ \sigma_{+} = .0002561.k, \end{array} \right. \quad \text{for } \left| x_3 : \begin{array}{l} \sigma_{-} = .0003316.k, \\ \sigma_{+} = .0000936.k, \end{array} \right. \tag{4.19}$$

k is the Gaussian constant.

Let us assume now that the quantities x_2 and x_3 have the following distribution functions (differential distribution laws):

1) If $x_2 < x_{2av}$, then

$$\varphi(x_2) = \frac{1}{\sigma_1\sqrt{2\pi}} \cdot e^{-\frac{(x_2 - x_{2cp})^2}{2\sigma_-^2}} \tag{4.20}$$

2) If $x_2 > x_{2av}$, then

$$\varphi(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}} \cdot e^{-\frac{(x_2 - x_{2cp})^2}{2\sigma_+^2}}, \tag{4.21}$$

where σ_1 and σ_2 are normalization coefficients, determined from supplementary conditions.

(These conditions are as follows: a) It is necessary to have $\int_{-\infty}^{+\infty} \varphi(x)dx = 1$, b) the probability that $x_2 < x_{2av}$ (or $x_2 > x_{2av}$) should be determined by Table 10.)

Similar distribution functions can be introduced also for x_3. Then we can state, with a probability 99.73 percent, that x_2 will always be less than $x_{2av} + 3\sigma_+$ and more than $x_{2av} - 3\sigma_-$. An analogous statement holds also for x_3.

The last circumstance enables us to estimate, to a certain degree, the quantities $\overline{x}_2 - x_{2av}$ and $x_{2av} - \underline{x}_2$. Namely, we assume that

$$\overline{x}_2 - x_{2cp} = 3\sigma_+, \quad x_{2cp} - \underline{x}_2 = 3\sigma_-. \tag{4.22}$$

Similarly for x_3:

89

$$\overline{x}_3 - x_{3cp} = 3\sigma_+,\ x_{3cp} - \underline{x}_3 = 3\sigma_-, \tag{4.23}$$

σ_+ and σ_- are taken here for x_3. We then obtain for $\delta\tilde{x}_2$ and $\delta\tilde{x}_3$

$$\delta\tilde{x}_2 = -.0000414,\ \delta\tilde{x}_3 = -.0000407. \tag{4.24}$$

We can therefore obtain also a numerical expression for the quantities in (4.14). However, as will be shown later, the values of these quantities with the exception of 3, are of no use to us. We shall therefore not give the numerical values of these quantities.

Let us examine now how one obtains the average values of $A^{(i)}e$, $A_1^{(i)}e$, $A_2^{(i)}e$, αe, and their derivatives with respect to λ. It can be easily shown that the averaged values of these quantities cannot be obtained from formulas analogous to (4.14), since the discarded quantities can in this case be arbitrarily large, for the series for these quantities in powers of $x_2 - x_{2av}$ and $x_3 - x_{3av}$ are divergent. However, this difficulty can be overcome by using the so-called combined expansion, i.e., we retain for the quantities $A^{(i)}$, $A_1^{(i)}$, $A_2^{(i)}$, and the expansion in powers of $x_2 - x_{2av}$ and $x_3 - x_{3av}$, while for the quantity e we employ the absolutely convergent expansion

90

$$e - e_{cp} + \frac{\partial e}{\partial \sqrt{x_2}}(\sqrt{x_2} - \sqrt{x_{2cp}}) + \ldots, \qquad (4.25)$$

where

$$e = \sqrt{\frac{2x_2}{x_1} - \frac{x_2^2}{x_1^2}}; \; e_{cp} = \sqrt{\frac{2x_{2cp}}{\lambda - l_2x_{2cp} - l_3x_{3cp}} - \frac{x_{2cp}^2}{(\lambda - l_2x_{2cp} - l_3x_{3cp})^2}}, \qquad (4.26)$$

and the derivative $\frac{\partial e}{\partial \sqrt{x_2}}$ is calculated under the assumption that x_1 is replaced by $\lambda - l_2x_2 - l_3x_3$. This derivative is given with sufficient accuracy by the formula

$$\frac{\partial e}{\partial \sqrt{x_2}} = \sqrt{\frac{2}{\lambda - l_2x_{2cp} - l_3x_{3cp}}}. \qquad (4.27)$$

We note that the similar derivative $\frac{\partial e}{\partial \sqrt{x_3}}$ is small and we neglect it.

The average values of $A^{(i)}e$, $\overline{A_1^{(i)}}e$, αe, and their derivatives with respect to λ are then given by the formulas:

$$\left.\begin{aligned} &\overline{\overline{A^{(i)}e}} = A^{(i)}\left(e_{cp} + \frac{\partial e}{\partial \sqrt{x_2}} \cdot \delta\sqrt{x_2}\right) + e_{cp}\frac{\partial A^{(i)}}{\partial \tilde{x}_2} \cdot \delta\tilde{x}_2 + e_{cp}\frac{\partial A^{(i)}}{\partial \tilde{x}_3} \cdot \delta\tilde{x}_3, \\ &\text{similarly for } \overline{\overline{A_1^{(i)}e}}, \\ &\overline{\overline{\alpha e}} = \alpha\left(e + \frac{\partial e}{\partial \sqrt{x_2}} \cdot \delta\sqrt{x_2}\right) + \frac{\partial \alpha}{\partial \tilde{x}_2} \cdot e_{cp} \cdot \delta\tilde{x}_2 + \frac{\partial \alpha}{\partial \tilde{x}_3} \cdot e_{cp} \cdot \delta\tilde{x}_3, \end{aligned}\right\} \qquad (4.28)$$

$$\left.\begin{aligned}
&\frac{\partial}{\partial\lambda}\left(\overline{\overline{A^{(i)}e}}\right)=e\frac{\partial A^{(i)}}{\partial\lambda}+A^{(i)}\frac{\partial e}{\partial\lambda}+\left(e\frac{\partial^2 A^{(i)}}{\partial\widetilde{x}_2\partial\lambda}+\frac{\partial A^{(i)}}{\partial\widetilde{x}^2}\cdot\frac{\partial e}{\partial\widetilde{x}_2}\right)\cdot\delta\widetilde{x}_2+\\
&+\left(e\frac{\partial^2 A^{(i)}}{\partial\widetilde{x}_3\partial\lambda}+\frac{\partial A^{(i)}}{\partial\lambda}\cdot\frac{\partial e}{\partial\widetilde{x}_3}\right)\cdot\delta\widetilde{x}_3+\left(\frac{\partial A^{(i)}}{\partial\lambda}\cdot\frac{\partial e}{\partial\sqrt{x_2}}+A^{(i)}\frac{\partial^2 e}{\partial\sqrt{x_2}\partial\lambda}\right)\cdot\delta\sqrt{x_2},\\
&\text{similarly for } \frac{\partial}{\partial\lambda}\overline{\overline{(A_1^{(i)}e)}},\\
&\frac{\partial}{\partial\lambda}\overline{\overline{(\alpha e)}}=\alpha\frac{\partial e}{\partial\lambda}+e\frac{\partial\alpha}{\partial\lambda}+\left(e\frac{\partial^2\alpha}{\partial\lambda\partial\widetilde{x}_2}+\frac{\partial e}{\partial\lambda}\cdot\frac{\partial\alpha}{\partial\widetilde{x}_2}\right)\cdot\delta\widetilde{x}_2+\\
&+\left(e\frac{\partial^2\alpha}{\partial\lambda\partial\widetilde{x}_3}+\frac{\partial e}{\partial\lambda}\cdot\frac{\partial\alpha}{\partial\widetilde{x}_3}\right)\cdot\delta\widetilde{x}_3+\left(\alpha\frac{\partial^2 e}{\partial\lambda\partial\sqrt{x_2}}+\frac{\partial\alpha}{\partial\lambda}\cdot\frac{\partial e}{\partial\sqrt{x_2}}\right)\cdot\delta\sqrt{x_2},
\end{aligned}\right\}\quad(4.29)$$

where

$$\delta\sqrt{x_2}=\frac{2}{3}\frac{\overline{x}_2+\sqrt{\overline{x}_2}\sqrt{\underline{x}_2}+\underline{x}_2}{\sqrt{\overline{x}_2}+\sqrt{\underline{x}_2}}-\sqrt{x_{2cp}},\qquad(4.30)$$

$$e\frac{\partial e}{\partial\lambda}=\left(1-\frac{x_2}{x_1}\right)\cdot\frac{x_2}{x_1^2},\quad\frac{\partial^2 e}{\partial\lambda\partial\sqrt{x_2}}=-\frac{1}{\sqrt{2}\,x_1^{3/2}},\qquad(4.31)$$

The remaining quantities are given by formulas (4.15) and (4.16). If we take into consideration our preceding calculations, we obtain $\delta\sqrt{x_2} = -.9723 \times 10^{-4}$.

Then

$$e_{av}+\frac{\partial e}{\partial\sqrt{x_2}}\cdot\delta\sqrt{x_2}=.0782764,\qquad(4.32)$$

whereas e_{av} =.079089, i.e., as a result of the averaging the value of the eccentricity, involved in the perturbation function, decreases, as it were.

As regards the quantities $A^{(i)}e^2$, $A_1^{(i)}e^2$, $A_2^{(i)}e$, $A_2^{(i)}e^2$, $B^{(i)}\eta^2$, αe^2, $\alpha\eta^2$, and their derivatives with respect to λ, the additions to these quantities, resulting from averaging, are found to be so small, that

92

they cannot influence the result of the calculation of the perturbations in our canonical variables, since these are in themselves small, or else are multiplied by the small quantities e_j or e_j^2.

After these calculations we can use formulas (2.52) and (2.53) of Chapter II to find the numerical values of the coefficients of the cosines, which we denoted in II by $K_n^{(i)}$, and their derivative with respect to λ.

4. Averaging over the Variable y_3 of the Perturbation Function and Its Partial Derivative With Respect to

Our perturbation function, as was noted in Chapter II, is an almost periodic function of y_3, if we replace in it

$$y_1 = \mu_1 - m_3^{(1)} y_3 = \mu_1 - y_3', \quad y_2 = \mu_2 - m_3^{(2)} y_3 = \mu_2 - \frac{m_3^{(2)}}{m_3^{(1)}} \cdot y_3', \tag{4.33}$$

and the basis of this almost-periodic function in $y_3' = m_3^{(1)} y_3$ is 1, $1/m_3^{(1)}$, and $m_3^{(2)}/m_3^{(1)}$.

Let us see now whether we can, starting from the specific form of the perturbation function of the problem (2.52) and (2.53) (Chapter II), see to it that by averaging within suitable limits over y_3' we simplify somewhat the form of the averaged perturbation function. It

93

turns out that this is possible. Namely, in the case $x_2 = x_3 = 0$, but $e_j \neq 0$, we obtain, after making the substitution (2.29), a perturbation function which is a periodic function of y_3' with period 2π. If we now carry out averaging over y_3' within such limit so as to make (see Chapter II) $\overline{y}_3 - \underline{y}_3'$ a multiple of 2π, then we obtain

$$\widetilde{\overline{\overline{(1)}}}^{(i)} = 0,\ \widetilde{\overline{\overline{(4)}}}^{(i)} = 0,\ \widetilde{\overline{\overline{(7)}}}^{(i)} = 0,\ \widetilde{\overline{\overline{(11)}}} = 0,\ \widetilde{\overline{\overline{(41)}}} = 0,\ \widetilde{\overline{\overline{(71)}}} = 0,\ \widetilde{\overline{\overline{(72)}}} = 0 \quad (4.34)$$

(i runs through all the integral values), with the exception of the case when $i = 0$ (two superior bars denote the averaging of the corresponding quantities with respect to x_2 and x_3, while the tilde denotes averaging with respect to y_3). A similar picture occurs also in the case of the derivatives of (4.34) with respect to λ.

We have assumed in our work that $\overline{y}_3$ (the upper limit of averaging with respect to y_3) amounts to $+ 2743.849^\circ$ (see Table 4), corresponding to $\overline{y}_3' = - 4322.9170^\circ$. If we assume $\underline{y}_3' = + 2157.0830^\circ$, then $\overline{y}_3' - \underline{y}_3' = - 6480.0000^\circ$, which precisely implies the equations (4.34). (Actually, the lower limit of the values of y_3 is $- 1368.424^\circ$ (see Table 4), corresponding to $y_3' = + 2155.9435^\circ$, which differs by only 1.1395° from the value of y_3' which we have assumed earlier.)

5. Determination of the Functions $\frac{\partial\Omega}{\partial\mu_1}, \frac{\partial\Omega}{\partial\mu_2}, \frac{\partial\Omega}{\partial\lambda}$

The procedure described above of averaging the perturbation function enables us now to find in explicit form the functions $\partial\Omega/\partial\mu_1$, $\partial\Omega/\partial\mu_2$, and $\partial\Omega/\partial\lambda$. These functions, as can be seen from formulas (2.83) and (2.84) of Chapter II, determine the time dependence of the variables $x_1, x_2, x_3; y_1, y_2, y_3$. The procedure for compiling these functions has been illustrated quite completely in Chapter II (Section 10) and we shall therefore not discuss it here. We note merely that for convenience in calculations we have, first, accounted for the constant terms in (2), (6), (21), and (61), i.e., the terms containing $M_{n0}^{(i)}$, by including them in the arbitrary integration constants, and second that the differences in the sines and cosines have been replaced, for convenience in calculation, by the products in accordance with the formulas

$$\left.\begin{aligned} &\sin(M_n^{(i)} + \varkappa_n^{(i)}\overline{y_3'}) - \sin(M_n^{(i)} + \varkappa_n^{(i)}\underline{y_3'}) = \\ &= 2\cos\left[M_n^{(i)} + \tfrac{1}{2}\varkappa_n^{(i)}(\overline{y_3'} + \underline{y_3'})\right]\cdot\sin\tfrac{1}{2}\varkappa_n^{(i)}(\overline{y_3'} - \underline{y_3'}), \\ &-\cos(M_n^{(i)} + \varkappa_n^{(i)}\overline{y_3'}) + \cos(M_n^{(i)} + \varkappa_n^{(i)}\underline{y_3'}) = \\ &= 2\sin\left[M_n^{(i)} + \tfrac{1}{2}\varkappa_n^{(i)}(\overline{y_3'} + \underline{y_3'})\right]\cdot\sin\tfrac{1}{2}\varkappa_n^{(i)}(\overline{y_3'} - \underline{y_3'}). \end{aligned}\right\} \quad (4.35)$$

The specific numerical values of the quantities $\varkappa_n^{(i)}$, $M_n^{(i)} + (1/2)\varkappa_n^{(i)}(\overline{y_3'} + \underline{y_3'})$, and $(1/2)\varkappa_n^{(i)}(\overline{y_3'} - \underline{y_3'})$ are

listed in Table 5.

In Table 7 of the same paper are given the secular terms, provided by the functions

$$\frac{\partial}{\partial\lambda}\int_{t=t_0}^{t} k^2 m_j \overset{\approx}{\overline{W}}_j\, dt,\quad \frac{\partial}{\partial\mu_1}\int_{t=t_0}^{t} k m_j \overset{\approx}{\overline{W}}_j\, dt,\quad \frac{\partial}{\partial\mu_2}\int_{t=t_0}^{t} k m_j \overset{\approx}{\overline{W}}_j\, dt,$$

and the corresponding periodic terms. These tables thus give the perturbations of Ceres relative to Jupiter, which are obtained by employing the orthointerpolational-averaged scheme of the limited three-dimensional elliptical three-point problem, since:

$$\frac{1}{k}\frac{\partial\Omega}{\partial\mu_1} = \frac{\partial}{\partial\mu_1}\int_{t=t_0}^{t} k m_j \overset{\approx}{\overline{W}}_j\, dt,\quad \frac{1}{k}\frac{\partial\Omega}{\partial\mu_2} = \frac{\partial}{\partial\mu_2}\int_{t=t_0}^{t} k m_j \overset{\approx}{\overline{W}}_j\, dt, \tag{4.36}$$

and the periodic perturbations in $\frac{\partial\Omega}{\partial\lambda}$ and $\int_{t=t_0}^{t} k^2 m_j \overset{\approx}{W}_j dt$ coincide. As regards the secular term in $\partial\Omega/\partial\lambda$, it will be determined in the next section by a different method.

6. Comparison of the Interpolative-analytical Theory of the Motion of Ceres with the Observations

As the initial values of the variables x_{1n}, x_{2n}, x_{3n}; y_{1n}, y_{2n}, y_{3n} we take the averaged values of the preliminary elements, bearing in mind that later on, after comparison with observation, we shall be able to

96

correct them. As the initial time reference we have taken the epoch 1850, January, 0. 0. As the epoch of the initial values of the preliminary elements we take the average epoch $\bar{t}$ of the preliminary elements, with, as it turns out,

$$(\bar{t} - 1850.\ \mathrm{I}.\ 0.\ 0)^{d} = +18200^{d}.069. \tag{4,37}$$

As regards the secular term N in the function $\partial\Omega/\partial\lambda$ we have assumed that it is equal to

$$N = \bar{n} - n_j, \tag{4.38}$$

where

$$\bar{n} = \frac{k^4}{(\lambda - l_2 x_{2cp} - l_3 x_{3cp})^3}. \tag{4.39}$$

Then

$$N'' = 471''.52358. \tag{4.40}$$

In this case we obtain for the epoch 1850, I. 0. 0 the following initial values of our canonical variables

$$\left.\begin{aligned} \sqrt{a} &= \frac{1}{k}\cdot x_{1n} = 1.663711, & y_{1n} &= 149^{\circ}.7124, \\ \sqrt{a} - \sqrt{p} &= \frac{1}{k}\cdot x_{2n} = .0052544, & y_{2n} &= 12.3552, \\ \sqrt{p} - \sqrt{p}\cdot\cos\gamma &= \frac{1}{k}\cdot x_{3n} = .0218881, & y_{3n} &= 81.7756. \end{aligned}\right\} \tag{4.41}$$

Taking these values into consideration, we can com-

pare the theory that we have constructed for the motion of Ceres with the observations.

For the comparison we have chosen eight normal places from 1801 through 1937. These normal places and the results of the comparison of the theory with the observations are given in Table 8 of page 89 [of source]. As can be seen from this table, the deviations in the direct ascension reach 2720". In the declination, the deviations are smaller and reach 1667". These deviations have the same order of magnitude as the deviations of the values of α and δ, calculated with the unperturbed elements, from the observed values of α and δ.

7. Correction of the Initial Values of the Canonical Variables. Repeated Comparison with Observations

After the first comparison of the theory with the observations, which gave such an unsatisfactory agreement, we have assumed that the discrepancy between the theory and observations can be attributed to the incorrect initial values of the canonical variables. We have corrected the initial values in the following manner.

First, in correcting the initial values we have sought now the correction not for the elements x_{1n}, x_{2n},

x_{3n}; y_{1n}, y_{2n}, y_{3n}, but the corresponding initial values of the Kepler elements

$$\left.\begin{array}{ll} M_0 = 160^\circ\!.9951 & i_e = 10^\circ 33' 29''\!.8, \\ \omega_e = 66^\circ 57' 7''\!.0, & n = 770''\!.65196, \\ \Omega_e = 81\ 59\ 41.1, & \varphi = 4^\circ 33' 17''\!.4, \\ & \text{epoch } 1850.I.0.0. \end{array}\right\} \quad (4.42)$$

Here ω_e, Ω_e, and i_e are referred to the ecliptic and to the equinox of 1950.0.

Second, in this correction we made use of these derivatives of α and δ with respect to the Kepler elements, which we obtained in Chapter III.

In processing the material of the initial comparison of the theory with the observations (see Table 8) we obtain first 16 conditional equations with six unknowns. Solving these equations by the least squares method, we obtained the following corrections to the initial values of the Kepler elements:

$$\left.\begin{array}{ll} \Delta M_0 = -40' 17''\!.1, & \Delta n = +''\!.03014582, \\ \Delta\omega_e = +\ 6\ 42.7, & \Delta\varphi = +10' 38''\!.4, \\ \Delta\Omega_e = +24\ 45.9, & \Delta i_e = -\ 2\ 48.0. \end{array}\right\} \quad (4.43)$$

We now find the corrected initial values of the canonical variables. We obtain:

$$\left.\begin{array}{ll} \sqrt{a} = \frac{x_{1n}}{k} = 1.6634916, & y_{1n} = 149^\circ\!.5670, \\ \sqrt{a} - \sqrt{p} = \frac{x_{2n}}{k} = .00567041, & y_{2n} = 11.8292, \\ \sqrt{p} - \sqrt{p}\cos\gamma = \frac{x_{3n}}{k} = .00213397, & y_{n3} = 81.3175. \end{array}\right\} \quad (4.44)$$

In addition

$$N^{0} = {}^{\circ}.1309871461. \tag{4.45}$$

The quantity $\sqrt{a}$ in (4.44) was obtained from that value of N, in which we take account of the secular part of $\frac{\partial}{\partial\lambda}\int_{t=t_0}^{t} k^2 m_j \tilde{\tilde{W}}_j dt$ (see Table 6).

The values of ℓ_2 and ℓ_3 for this value of N^0 are respectively + .63360174 and + .63467837.

After this correction of the initial values, we again compared the theory with the observation (we neglected the changes in the perturbations due to the correction of the initial values and of N^0, ℓ_2, and ℓ_3). The result of the comparison is given in Table 9.

In Table 10 we give a comparison with V. F. Proskurin's theory [14] for the motion of Ceres, derived by Hill's method (we have borrowed this comparison from the work of V. F. Proskurin and T. I. Mashinskaya [4]). We see that our theory agrees with the observations of the direct ascension just as well as Proskurin's theory. As to the declination, our discrepancies are found to be

approximately three times greater than those of Proskurin [4].

CONCLUSION

Principal conclusions of our work can be briefly formulated as follows:

1. Modern determinations of the positions of small planets are carried out with an error on the order of 1" in inclination and 2--3" in direct ascension.

2. None of the existing theories of the perturbed motion of Ceres present the observations with this degree of accuracy.

3. All the existing analytic and semi-analytic theories of the perturbed motion of Ceres are limited to an account of first-order perturbations relative to the perturbing masses only, although in the best theories the first-order perturbations of almost all the large planets (from Mercury to Neptune) were taken into account.

4. The better of the existing analytical and semi-analytical theories of the motion of Ceres present the observations in the time interval covered with an error on the order of 36" in inclination and 97" in direct ascension.

5. When the Hill method is used to calculate the

first-order perturbations in the motion of Ceres, much fewer significant figures are calculated reliably than are written out in the result.

6. In the tables of motion of the sun, compiled by Newcomb, there are errors and misprints, not noted during the publication of these tables.

7. The interpolation-analytic theory of perturbed motion of an asteroid, developed in general form in Chapter II of the present paper on the basis of the orthointerpolational triply-averaged variant of the limited three-dimensional elliptical three-point problem (sun, Jupiter, asteroid) can be successfully used also to construct an approximate intermediate theory of the perturbed motion of Ceres.

8. From the theoretical point of view, such a theory has the following main advantages over the corresponding theories of classical celestial mechanics: a) The basic most appreciable perturbations of the secular type over the time interval covered (interpolational-secular perturbations) are determined directly from the processing of the entire observational material and are determined with high degree of accuracy; b) within the limits of the perturbation-function terms accounted for in the interpolational-averaged scheme,

this theory takes direct account not only of the first-order perturbation, as takes place in the existing theories of the motion of Ceres, but of any order; c) the coefficients of the periodic perturbations are calculated in this theory by means of such formulas which involve no loss in accuracy as would occur, for example, if the Hill method were used; d) the general solution of the differential equations of perturbed motion is obtained in this theory in such a form as to permit a much more effective calculation of the residual terms of the employed infinite series, and consequently, makes it possible to estimate the error due to replacing these series with finite sums, i.e., an estimate which can hardly be obtained in the classical theories; e) within the framework of our interpolational-analytical theory we can, without increasing the volume of calculation, greatly decrease the rough nature of the classical method of correcting for the initial values of the osculating elements.

9. Bearing in mind a specific determination of only the approximate intermediate theory of perturbed motion of Ceres, we confined ourselves in the specific example of the application of the interpolational-analytical theory presented in our article only to the

use of part of the observational material, and confined ourselves in the perturbation function only to an account of terms of order not higher than the second relative to the eccentricities and the mutual inclination.

10. In spite of this, the approximate intermediate theory the we obtained for the motion of Ceres presented the employed observations with a mean square error of 98" in direct ascension and 115" in declination. This error is somewhat higher than the error in the better of the existing analytical theories of the motion of Ceres. However, in the sense of deviation along the great-circle arc, our error exceeds the greatest error of the preceding theories by only one and a half times. If we recognize that the error in the existing theories exceeds the error of the astrometic determinations of the positions by not less than forty times, it must be recognized that the approximate intermediate theory that we have obtained for the perturbed motion of Ceres is satisfactory.

11. To improve the interpolational-analytical theory of the motion of Ceres we can in the future do the following:

a) Construct an interpolational-analytical theory of the motion of Ceres with corrected initial values of the osculating elements, taking into account all the influ-

ential terms in the perturbation function and find an estimate for the magnitude of the discarded terms.

b) Obtain additional corrections for the initial values of the osculating elements on the basis of the newly devised theory. In this correction better values should be obtained for the coefficients in the interpolational elements. In addition, to obtain this correction we must use more extensive observational material.

All this can be done without principal difficulties.

BIBLIOGRAPHY

1. Hill, G. Astron. J., 21, 51, 1900. Works, 4, 212, 1907.

2. Commendantoff N., Byul Ast inst (Bulletin of Astronomic Institute), 1, No 2, 10, 1924; No 14, 163, 1926.

3. Rabe, E. Astron. Nachr. 270, No 2, 82-93, 1940.

4. Proskurin, V. F. and Mashinskaya T. I., Byul ITA AN SSSR (Bulletin, Institute of Theoretical Astronomy, Academy of Sciences, USSR), No 5 (68), V, 315, 1953.

5. Orlov A. Ya and Orlov B. A. Kurs teoreticheskoy astronomii (Course of Theretical Astronomy), GITTL, Moscow-Leningrad, 1940.

6. Struve, O. Tabulae quantitatum besselianarum... 1840 and 1864, Petropoli 1861. 1750-1840.

7. Newcomb S. Astron. Papers. Vol VI. Wash. 1898; Astron. Papers. V. XIV. Wash. 1954.

8. Commendantov, N. M. Izv GAI (News of the Main Astronomical Observatory) 12, No 197, 1, 1930.

9. Shteins K., Use of Averaged Variants of the Three-point Problem as Applied to the Theory of Small Planets (Candidate's dissertation of K. A. Shtein).

10. Moiseyev, N. D. Vestnik MGU (Herald, Moscow State University) No 2, 29, 1950.

11. Moiseyev, N. D., Tr GAISH (Works of the Shternberg Astronomic Observatory), Vol 24, 3, 1954.

12. Moiseyev, N. D. ibid, Vol 24, 10, 1954.

13. Moiseyev, N. D. On the Orthointerpolational Averaged Version of the Bounded Circular Three-point Problem, see page 9 of the present volume.

14. V. F. Proskurin, Tr ITA AN SSSR, No 2, 1952.

15. M. F. Subbotin, Kurs nebesnoy mekhaniki (Course of Celestial Mechanics), Vol 2, ONTI, 1937.

16. Le Verrier U. Annales de l'observatoie de Paris, 1, 1855.

17. Samoilova-Yakhontova N. S., Byul Ast Inst, 1932 No 46.

18. Bohl P. Ueber die Darstellung von Funktionen einer Variablen durch trigonometrische Reihen mit mehreren einer Variablen proportionalen Argument (Master's Dissert.), Dorpat, 1893, 31 pp.

19. Yarov-Yarovoi M. S., Tr GAISH. Vol 24, 17, 41, 1954.

20. M. F. Subbotin, Kurs Nebestnoi mekhaniki, Vol 1, OGIZ, 1941.

Table 1.

Normal Places of Ceres, Obtained by Hill [1].

a) № норм. мест	б) Эпоха. Средний гриничский полдень	$\alpha_{1950.0}$	$\delta_{1950.0}$
1	1801 янв. (1) 21	53°44′31″.6	+17°25′25″.4
2	1802 март (3) 31	182 37 21.7	+17 10 28.7
3	1803 июль (7) 8	281 35 25.7	−28 35 24.0
4	1804 окт. (10) 4	11 12 59.9	−12 6 18.0
5	1806 янв. (1) 21	102 47 39.9	+30 24 46.4
6	1807 май (5) 6	224 30 17.9	− 5 52 55.8
7	1808 авг. (8) 10	320 53 25.2	−28 49 58.5
8	1809 ноябрь (11) 3	43 19 43.7	+ 5 29 34.4
9	1811 февр. (2) 24	159 23 40.8	+26 17 36.2
10	1812 июнь (6) 12	261 49 9.6	−23 15 25.2
11	1813 сент. (9) 11	355 46 22.4	−19 24 53.9
12	1816 апр. (4) 12	203 16 33.4	+ 6 29 17.8
13	1818 окт. (10) 13	26 56 18.0	− 3 53 42.9
14	1820 февр. (2) 7	129 12 18.6	+31 29 51.4
15	1821 май (5) 25	241 25 54.3	−15 14 8.7
16	1822 авг. (8) 24	338 19 58.2	−25 11 6.4
17	1823 ноябрь (11) 24	59 11 19.5	+14 10 47.9
18	1825 март (3) 19	182 14 46.3	+17 44 57.2
19	1826 июнь (6) 28	281 14 19.5	−27 48 17.0
20	1827 сент. (9) 26	10 41 20.4	−12 29 1.9
21	1829 янв. (1) 30	98 0 41.4	+30 35 51.4
22	1830 май (5) 1	222 26 40.1	− 4 38 59.8
23	1831 июль (7) 29	321 17 19.2	−28 5 51.7
24	1832 окт. (10) 31	41 46 14.2	+ 4 33 42.0
25	1834 февр (2) 17	157 0 0.4	+26 42 15.8
26	1835 июль (7) 10	260 15 26.9	−22 42 33.4
27	1836 сент. (9) 10	354 0 9.3	−19 0 23.1
28	1837 дек. (12) 15	77 5 4.2	+22 25 21.9
29	1839 апр. (4) 13	200 53 30.5	+ 7 37 44.1
30	1840 авг. (8) 2	296 24 34.4	−30 59 38.4
31	1841 окт. (10) 26	22 14 48.1	− 5 17 23.9
32	1843 февр. (2) 5	126 49 24.2	+31 41 43.9
33	1844 май (5) 17	240 59 22.4	−14 28 10.3
34	1845 авг. (8) 29	335 36 1.4	−26 6 48.2
35	1846 ноябрь (11) 19	58 19 59.9	+13 28 37.1
36	1848 март (3) 28	177 27 50.4	+19 25 53.7
37	1849 июль (7) 5	277 11 46.9	−28 8 9.7
38	1850 сент. (9) 18	10 44 15.2	−12 34 36.2
39	1852 янв. (1) 8	99 45 31.7	+29 4 26.4
40	1853 май (5) 5	218 56 41.6	− 3 27 6.7
41	1854 авг. (8) 13	315 55 34.5	−29 52 40.4
42	1855 ноябрь (11) 4	38 47 45.5	+ 3 32 59.5
43	1857 февр. (2) 22	153 10 40.6	+27 59 30.2
44	1858 июнь (6) 6	259 15 31.0	−22 8 30.8
45	1859 сент. (9) 1	354 10 18.4	−19 43 41.7
46	1860 дек. (12) 4	77 27 12.0	+21 21 43.4
47	1862 апр. (4) 21	197 27 54.0	+ 8 37 30.6

Table 1 (continued).

№ норм. мест	Эпоха. Средний гринвичский полдень	$\alpha_{1950.0}$	$\delta_{1950.0}$
48	1863 июль 7 16	298°33′ 8″.8	−29°50′42″.2
49	1864 окт. 10 16	22 40 4.7	− 5 42 21.7
50	1866 февр. 2 4	124 0 7.7	+31 50 43.4
51	1867 май 5 26	237 34 5.3	−43 58 33.4
52	1868 авг. 8 27	334 18 58.0	−26 26 10.5
53	1869 ноябрь 11 23	55 13 11.8	+12 50 52.1
54	1871 март 3 19	176 46 25.5	+20 4 13.3
55	1872 июнь 6 30	276 16 21.6	−27 41 58.3
56	1873 сент. 9 30	6 44 51.1	−14 9 5.8
57	1875 янв. 1 5	97 28 17.5	+28 37 30.4
58	1876 май 5 6	216 23 54.1	− 2 22 36.6
59	1877 авг. 8 17	313 9 58.7	−30 28 14.7
60	1878 ноябрь 11 7	36 27 17.1	+ 2 44 55.4
61	1880 февр. 2 28	149 27 0.2	+29 3 20.0
62	1881 июль 7 16	254 32 26.4	−22 2 1 4
63	1882 сент. 9 23	347 51 34.8	−22 6 51.4
64	1883 дек. 12 4	75 35 46.0	+20 51 27.5
65	1885 апр. 4 14	197 40 18.0	+ 9 53 10.0
66	1886 июль 7 30	292 48 4.2	−30 54 57.8
67	1887 окт. 10 22	19 51 8.5	− 6 44 38.4
68	1889 февр. 2 4	121 8 12.9	+32 2 52.3
69	1890 май 5 27	234 46 43.6	−13 3 11.9
70	1891 сент. 9 6	330 16 9.7	−27 35 43.7
71	1892 ноябрь 11 18	54 6 59.7	+11 56 59.7
72	1894 март 3 15	174 13 36.7	+21 11 7.2
73	1895 июнь 6 22	276 5 38.2	−26 58 21.6
74	1896 сент. 9 26	5 46 35.9	−14 40 23.3
75	1897 дек. 12 19	98 18 4.5	+26 59 55.1

Key: a) Number of normal places; b) Epoch, Mean GMT.

1) January
2) February
3) March
4) April
5) May
6) June
7) July
8) August
9) September
10) October
11) November
12) December

Table 2.

Normal Places of Ceres, Obtained by N. V. Komendantov, E. Rabe, and V. F. Proskurin.

a № норм. мест	b Эпохи нормальных мест				$\alpha_{1950.0}$	$\delta_{1950.0}$	c Литерат. источник
1	1920	окт.	10	3.5	96°56′11″.4	+21°59′55″.3	[2]
2		окт.	10	27.5	100 36 44.8	+22 51 39.9	»
3		ноябрь	11	24.5	100 9 1.0	+24 35 25.0	»
4		дек.	12	22.5	94 24 44.9	+26 51 4.7	»
5	1921	янв.	1	7.5	90 16 36.9	+27 55 14.8	»
6		янв.	1	23.5	87 6 16.8	+28 39 14.0	»
7		февр.	2	16.5	85 52 22.1	+29 17 50.8	»
8		март	3	8.5	88 15 4.8	+29 36 47.8	»
9		апр.	4	9.5	96 56 38.9	+29 41 1.6	»
10		апр.	4	29.5	104 24 32.9	+29 17 9.4	»
11	1922	апр.	4	4.5	219° 2′50″.3	− 1°26′10″.8	»
12		апр.	4	20.5	215 51 47.8	− 0 40 59.8	»
13		май	5	2.5	213 10 53.4	− 0 23 27.7	»
14		май	5	18.5	210 2 25.7	− 0 33 18.3	»
15		май	5	26.5	208 52 53.9	− 0 53 32.8	»
16	1923	март	3	14.5	300 40 46.4	−23 46 13.9	»
17		май	5	17.5	317 54 59.0	−23 10 22.2	»
18		июнь	6	26.5	319 28 21.8	−23 53 28.0	»
19		авг.	8	5.5	312 22 31.0	−30 7 34.7	»
20		сент.	9	18.5	305 45 18.8	−31 12 32.6	»
21		окт.	10	30.5	309 54 5.9	−28 44 53.4	»
22	1924	сент.	9	17.0	41 46 51.5	+ 3 32 24.1	»
23		окт.	10	19.0	37 13 29.5	+ 1 49 17.3	»
24		дек.	12	14.0	27 42 33.1	+ 2 15 0.9	»
25	1925	февр.	2	4.0	33 6 27.3	+ 8 14 22.1	»
26		окт.	10	19.0	137 34 13.1	+21 47 42.5	»
27	1926	февр.	2	8.0	147 59 1.9	+28 37 44.8	»
28		апр.	4	1.0	140 16 19.7	+29 56 43.5	»
29	1926	март	3	26.0	140 19 26.8	+30 16 59.6	[3]
30	1927	июнь	6	7.0	252 54 49.0	−20 38 30.2	»
31	1928	сент.	9	4.0	348 49 16.5	−21 54 20.4	»
32	1931	март	3	15.0	197 52 8.6	+10 1 7.7	»
33	1933	окт.	10	8.0	19 53 7.5	− 7 27 35.9	»
34	1935	янв.	1	22.0	117 56 0.1	+31 11 53.4	»
35	1936	март	3	20.0	239 43 2.1	−11 22 4.4	»
36		май	5	21.0	231 10 47.0	−10 57 25.1	[3]
37		июнь	6	19.0	225 43 15.3	−14 0 40.1	»
38	1937	сент.	9	16.0	324 50 7.0	−28 45 7.3	»
39	1938	ноябрь	11	27.5	47 49 14.1	+10 11 1.1	»
40	1906	дек.	12	6.98268	72 48 57.4	+20 23 5.2	[8]
41	1915	ноябрь	11	15.5	52 39 55.8	+10 58 5.7	»
42	1946	авг.	8	29.0	305 39 52.0	−31 30 18.3	[4]

Key: a) Number of normal places; b) Epochs of normal places; c) Bibliographic source.
[Key for months same as in Table 1.]

Table 3.

Values of Preliminary Elements of Ceres.

№	n	φ	$\Delta' M_0$	Δ' ☊	$\Delta' \omega$
1	799″.537	2 52′ 5″.4	+152°.151	+1°24′53″.1	−13 42′50″.0
2	770.984	4 41 8.3	+ 17.659	+0 45 2.9	− 4 33 17.0
3	770.777	4 34 13.9	+ 2.573	+0 45 36.7	− 3 9 31.0
4	787.569	3 42 57.2	+ 68.718	+1 1 30.7	+ 5 29 3.3
5	774.845	4 28 44.7	+ 23.197	+1 5 57.9	− 6 10 2.9
6	790.809	12 4 45.8	+105.906	−1 0 59.4	−33 2 31.5
7	792.195	4 43 55.7	+ 69.541	+1 4 34.0	+16 41 22.2
8	770.295	4 17 14.9	+ 4.845	−0 4 17.0	+ 3 43 13.1
9	770.798	4 35 59.5	+ .748	−0 12 12.5	− 0 20 23.0
10	770.783	4 35 40.0	− .467	−0 11 50.4	− 0 7 26.3
11	770.809	4 35 40.0	+ .433	−0°13′12″.7	− 0°49′ 2″.2
12	770.877	4 38 19.6	+ 9.503	−1 6 7.0	+ 4 53 12.1
13	770.613	4 30 22.6	− 1.189	−1 8 20.7	+ 3 31 43.7
14	770.717	4 33 32.4	− .735	−1 10 53.2	+ 2 35 43.8
15	770.748	4 34 36.8	− 1.149	−1 10 3.6	+ 2 44 18.8
16	770.288	4 34 51.9	− 26.262	−2 1 54.5	+ 1 11 48.0
17	771.002	4 48 4.9	− 18.364	−1 36 6.3	+ 6 42 16.3
18	770.416	4 27 29.2	− .735	−0 50 40.8	+ 2 23 34.3
19	770.433	4 24 32.1	+ .490	−1 14 34.6	+ 2 18 78.5
20	770.370	4 22 42.4	− .233	−1 15 30.9	+ 2 18 35.0
21	770.400	4 22 11.3	− 4.091	−1 11 31.7	+ 1 45 31.4
22	770.774	4 28 20.9	+ 2.203	−1 30 0.5	+ 3 24 21.7

№	$\Delta' i$	$(t_k' - 1850.I.0.0)^d$	№	$\Delta' i$	$(t_k' - 1850.I.0.0)^d$
1	+0° 4′59″.2	−17431.5000	12	−0°1′ 5″.4	+26403.5000
2	−0 0 3.9	16981.8333	13	−0 1 6.1	26869.0000
3	+0 0 20.2	16517.8333	14	−0 1 8.3	27332.2667
4	−0 4 28.0	16051.8333	15	−0 1 10.7	27799.3333
5	+0 1 56.2	15582.2667	16	−0 8 32.1	28272.0000
6	−0 10 40.2	15121.5000	17	−0 4 23.7	28891.6667
7	−0 4 52.8	−14658.2667	18	−0 2 33.4	29663.3333
8	+0 0 52.6	+ 4947.8333	19	−0 1 2.3	30440.3333
9	−0 0 39.5	5409.5000	20	−0 1 6.9	31071.6667
10	−0 0 43.4	5879.5000	21	−0 0 23.4	31551.3333
11	−0 0 21.8	6349.8333	22	+0 2 1.5	32019.8333

Table 4.

Preliminary Elements of Canonical Variables.

№*	M	M_j	y_1	y_2	y_3
1	−3558°.279	−1300°.372	−2133°.580	−1424°.699	−1308°.424
2	3470 .725	1263 .005	2074 .897	1395 .828	1330 .286
3	3372 .755	1224 .451	2014 .075	1358 .680	1291 .745
4	3281 .926	1185 .731	1953 .059	1328 .867	1253 .304
5	3169 .638	1146 .711	1891 .387	1278 .251	1214 .358
6	3054 .826	1108 .431	1843 .823	1211 .003	1173 .666
7	−2995 .062	−1069 .935	−1770 .734	−1224 .328	−1137 .565
8	+1216 .547	+ 559 .152	+ 797 .673	+ 418 .874	+ 492 .796
9	1319 .735	597 .515	858 .306	461 .429	531 .315
10	1420 .453	636 .568	919 .726	500 .427	570 .528
11	1519 .973	675 .646	979 .919	540 .054	609 .463
12	5809 .655	2341 .932	3608 .139	2201 .516	2276 .748
13	5910 .772	2380 .611	3669 .232	2241 .540	2315 .469
14	6010 .821	2419 .099	3729 .751	2281 .670	2354 .005
15	6110 .881	2457 .911	3791 .173	2319 .708	2392 .802
16	6214 .579	2497 .488	3853 .189	2361 .390	2433 .090
17	6340 .960	2548 .680	3933 .015	2407 .945	2485 .076
18	6510 .035	2612 .793	4035 .421	2474 .614	2547 .325
19	6676 .808	2677 .355	4137 .157	2539 .651	2612 .331
20	6811 .944	2729 .818	4219 .809	2592 .135	2664 .811
21	6915 .087	2769 .669	4282 .617	2632 .470	2704 .583
22	+7013 .825	+2808 .597	+4343 .766	+2670 .059	+2743 .849

№*	x_1/k	x_2/k	x_3/k	№*	x_1/k	x_2/k	x_3/k
	1.6	.0	.0		1.6	.0	.0
1	43322	020591	222802	12	63440	054488	221208
2	63363	055593	221292	13	63630	051426	221274
3	63512	052900	221664	14	63555	052634	221238
4	51604	034717	216479	15	63533	053047	221186
5	60596	050720	220907	16	63864	053156	215843
6	49345	365204	207886	17	63250	058370	218752
7	48383	056178	215471	18	63772	050402	220057
8	63859	046562	222521	19	63760	049238	221402
9	63497	053581	221222	20	63805	048558	221373
10	63508	053454	221171	21	63783	048364	221921
11	63489	053453	221462	22	63514	050656	223872

*) Numbers of systems of preliminary elements taken from Table 4 [sic !].

112

Table 5.

Values of $\varkappa_n^{(i)}$, $\overline{M}_n^{(i)}+\frac{1}{2}\varkappa_n^{(i)}(\overline{y}_3'+\underline{y}_3')$, $\frac{1}{2}\varkappa_n^{(i)}(\overline{y}_3'-\underline{y}_3')$

$$\varkappa_2^{(i)}=-1.633645-i,$$
$$\varkappa_3^{(i)}=+\ .366355-i,$$
$$\varkappa_5^{(i)}=+3.267290-i,$$
$$\varkappa_6^{(i)}=+1.633645-i,$$
$$\varkappa_8^{(i)}=+3.269447-i.$$

$$\overline{M}_2^{(i)}+\frac{1}{2}\varkappa_2^{(i)}(\overline{y}_3'+\underline{y}_3')=280^\circ.6958+23^\circ.7925.\,i,$$
$$M_3^{(i)}+\frac{1}{2}\varkappa_3^{(i)}(\overline{y}_3'+\underline{y}_3')=233\ .1108+23\ .7925.\,i,$$
$$\overline{M}_5^{(i)}+\frac{1}{2}\varkappa_5^{(i)}(\overline{y}_3'+\underline{y}_3')=158\ .6083+23.7925.\,i,$$
$$\overline{M}_6^{(i)}+\frac{1}{2}\varkappa_6^{(i)}(\overline{y}_3'+\underline{y}_3')=79\ .3042+23.7925.\,i,$$
$$\overline{M}_8^{(i)}+\frac{1}{2}\varkappa_8^{(i)}(\overline{y}_3'+\underline{y}_3')=17\ .7093+23.7925.\,i,$$

Values of $\varkappa_n^{(i)}$, $\overline{M}_n^{(i)}+\frac{1}{2}\varkappa_n^{(i)}(\overline{y}_3'+\underline{y}_3')$, $\frac{1}{2}\varkappa_n^{(i)}(\overline{y}_3'-\underline{y}_3')$

$$\frac{1}{2}\varkappa_2^{(i)}(\overline{y}_3'-\underline{y}_3')=253^\circ.0098,$$
$$\frac{1}{2}\varkappa_3^{(i)}(\overline{y}_3'-\underline{y}_3')=253.0098$$
$$\frac{1}{2}\varkappa_5^{(i)}(\overline{y}_3'-\underline{y}_3')=213.9804,$$
$$\frac{1}{2}\varkappa_6^{(i)}(\overline{y}_3'-\underline{y}_3')=--253^\circ.0098,$$
$$\frac{1}{2}\varkappa_8^{(i)}(\overline{y}_3'-\underline{y}_3')=206^\circ.9917.$$

113

Table 6.

Coordinates of the Sun for the Epochs of the Normal Places of J. Hill, N. V. Komendantov, and E. Rabe, Referred to the Equator and Equinox of 1950.0.

№*	$X_{1950.0}$	$Y_{1950.0}$	$Z_{1950.0}$
1	+.5382772	−.7559773	−.3281425
2	+.9776097	+.1926733	+.0836701
3	−.3029019	+.8902550	+.3864358
4	−.9846332	−.1640644	−.0698748
5	+.5341773	−.7584226	−.3291953
6	+.6886542	+.6771671	+.2939651
7	−.7714391	+.6027775	+.2616245
8	−.7290367	−.6159432	−.2673850
9	+.9114055	−.3553236	−.1542055
10	+.4202626	+.9253537	+.4016630
11	−.9912266	+.1567256	+.0679956
12	+.9143220	+.3790932	+.1645741
13	−.9281839	−.3335254	−.1447952
14	+.7503575	−.5876334	−.2550368
15	+.4173501	+.8472070	+.3677304
16	−.8961110	+.4282653	+.1858512
17	−.4446556	−.8080570	−.3507326
18	+.9964655	+.0057463	+.0025261
19	−.1396533	+.9238490	+.4009687
20	−.9990398	−.0691756	−.0300502
21	+.6608326	−.6706185	−.2910375
22	+.7463353	+.6219446	+.2699496
23	−.6139152	+.7415558	+.3218241
24	−.7626860	−.5817580	−.2525054
25	+.8565324	−.4530520	−.1966021
26	+.1686973	+.9186384	+.3986911
27	−.9888488	+.4705325	+.0739875
28	−.0864884	−.8990678	−.3901886
29	+.9136950	+.3805081	+.1651583
30	−.6749272	+.6948198	+.3015193
31	−.8191870	−.5152142	−.2236119
32	+.7282866	−.6104368	−.2647640
33	+.5347657	+.7881815	+.3420652
34	−.9318550	+.3557083	+.1543425
35	−.5195233	−.7706487	−.3344453
36	+.9856926	+.1492226	+.0647838
37	−.2575518	+.9022492	+.3915351
38	−1.0022805	+.0548837	+.0237933
39	+.3170970	−.8538637	−.3705282
40	+.6983135	+.6685612	+.2901353
41	−.7946788	+.5758958	+.2498887
42	−.7266448	−.6183927	−.2683590
43	+.8983187	−.3813461	−.1654634
44	+.2333853	+.9062829	+.3932733
45	−.9463743	+.3203000	+.1389638

*) Numbers of epochs taken from Table 1.

Table 6 (continued).

№*	$X_{1950.0}$	$Y_{1950.0}$	$Z_{1950.0}$
46	−.2736148	−.8681389	−.3767186
47	+.8497304	+.4933707	+.2141026
48	−.4232019	+.8477001	+.3678246
49	−.9057663	−.3799750	−.1648953
50	+.7175066	−.6206012	−.2692722
51	+.4140277	+.8486008	+.3682281
52	−.9194003	+.3829395	+.1661469
53	−.4570558	−.8024112	−.3481793
54	+.9961826	−.0069274	−.0029897
55	−.1767127	+.9185611	+.3985555
56	−.9897825	−.1350074	−.0585992
57	+.2687099	−.8677311	−.3764974
58	+.6846618	+.6805509	+.2952936
59	−.8353833	+.5239701	+.2273245
60	−.6883643	−.6532488	−.2834405
61	+.9329402	−.3065161	−.1329787
62	+.0643793	+.9302725	+.4036214
63	−1.0025371	−.0205887	−.0089472
64	−.2886645	−.8642299	−.3750723
65	+.9052354	+.3978663	+.1726331
66	−.6259972	+.7329428	+.3179833
67	−.8643309	−.4514800	−.1958859
68	+.7187840	−.6194143	−.2687267
69	+.3969592	+.8556340	+.3712257
70	−.9704613	+.2485409	+.1078168
71	−.5313617	−.7640551	−.3314864
72	+.9922573	−.0685477	−.0297232
73	−.0271772	+.9321542	+.4044048
74	−.9985573	−.0741067	−.0321588
75	−.0233750	−.9021984	−.3914018
1	−.9838099	−.1652437	−.0716851
2	−.8217853	−.5120241	−.2221036
3	−.4582291	−.8019437	−.3478592
4	−.0133553	−.9022289	−.3913574
5	+.2888631	−.8623390	−.3740496
6	+.5415294	−.7541135	−.3271690
7	+.8352864	−.4843884	−.2101086
8	+.9704977	−.1926344	−.0835503
9	+.9445236	+.3067684	+.1330706
10	+.7827080	+.5818427	+.2523903
11	+.9693139	+.2271186	+.0985183
12	+.8702550	+.4610648	+.2000002
13	+.7528970	+.6150251	+.2667760
14	+.5490068	+.7796371	+.3381823
15	+.4309772	+.8413176	+.3649313
16	+.9875485	−.1068979	−.0463593
17	+.5665589	+.7687422	+.3334496
18	−.0749648	+.9300827	+.4034335
19	−.6840651	+.6871822	+.2980626
20	−1.0007096	+.0817229	+.0354428

Table 6 (conclusion).

№*	$X_{1950.0}$	$Y_{1950.0}$	$Z_{1950.0}$
21	—.7990309	—.5406020	—.2344993
22	—.9996593	+.0936259	+.0406010
23	—.8973540	—.3959404	—.1717503
24	—.1353184	—.8943046	—.3879125
25	+.6975207	—.6391400	—.2772283
26	—.8993701	—.3923549	—.1701898
27	+.7424857	—.5958839	—.2584613
28	+.9814599	+.1726446	+.0748930
29	+.9939325	+.0785852	+.0340932
30	+.2526068	+.9018039	+.3911623
31	—.9562159	+.2934225	+.1272651
32	+.9885349	—.0998878	—.0433219
33	—.9671849	—.2292486	—.0994335
34	+.5105402	—.7718921	—.3347915
35	+.9960574	—.0094397	—.0040896
36	+.5070061	+.8038838	+.3486664
37	—.4505459	+.8357135	+.3624655
38	—.9974672	+.1143444	+.0495923
39	—.4234946	—.8175297	—.3545738

116

Table 7.

The Functions $k^2 m_j \int_{t=t_0}^{t} \frac{\partial}{\partial \lambda} (\widetilde{\omega}_j)\, dt,\ k m_j \int_{t=t_0}^{t} \frac{\partial}{\partial \mu_1} (\widetilde{\omega}_j)\, dt,\ k m_j \int_{t=t_0}^{t} \frac{\partial}{\partial \mu_2} (\widetilde{\omega}_j)\, dt$[1].

A. Table of Periodic Terms of the Function $k^2 m_j \int_{t=t_0}^{t} \frac{\partial}{\partial \lambda} (\widetilde{\omega}_j)\, dt$[2].

i	Sin [3]	Sin [4]	i	Sin [3]	Sin [4]
−20	− ″.004	+ ″.000	+20	+ ″.003	−.013″
19	.006	.001	19	.005	.023
18	.011	.002	18	.010	.038
17	.019	.004	17	.017	.067
16	.033	.007	16	.031	.117
15	.057	.013	15	.052	.203
14	.099	.025	14	.088	.350
13	.174	.046	13	.147	.604
12	.283	.069	12	.243	1.018
11	.471	.113	11	.398	1.720
10	.773	.180	10	.641	2.879
9	1.251	.300	9	1.019	4.742
8	2.039	.436	8	1.618	7.840
7	3.062	.606	7	2.362	12.829
6	4.636	.889	6	3.431	20.552
5	6.675	1.123	5	4.686	33.046
4	9.334	1.467	4	5.985	53.824
3	10.545	1.281	3	6.289	92.816
2	2.146	1.303	2	4.552	−283 .854
− 1	23.714	2.265	+1	7.123	+118 .370
0	−11 .568	+11 .568			

1) This expression includes also the indirect term of the perturbation function.

2) The function $k^2 m_j \int_{t=t_0}^{t} \frac{\partial}{\partial \lambda} (\widetilde{\omega}_j)\, dt$ contains also the periodic terms:
$+''.030 . \sin M_j + ''.003 \sin 2M_j$.

3) Coefficients of the sines and cosines $\overline{M}_2^{(i)} + \frac{1}{2} \varkappa_2^{(i)} (\overline{y}_3' + \underline{y}_3') - M_j$.

4) Coefficients of the sines and cosines $\overline{M}_6^{(i)} + \frac{1}{2} \varkappa_6^{(i)} (\overline{y}_3' + \underline{y}_3') - M_j$.

117

Table 7 (continued).

B. Table of Periodic Terms of the Functions $km_j \int\limits_{t=t_0}^{t} \frac{\partial}{\partial \mu_1} (\widetilde{\omega}_j)\, dt,\ km_j \int\limits_{t=t_0}^{t} \frac{\partial}{\partial \mu_2} (\widetilde{\omega}_j)\, dt$

i	Function $km_j \int\limits_{t=t_0}^{t} \frac{\partial}{\partial \mu_1} (\widetilde{\omega}_j)\, dt$		Function $km_j \int\limits_{t=t_0}^{t} \frac{\partial}{\partial \mu_2} (\widetilde{\omega}_j)\, dt$	
	Cos [1]	Cos [2]	Cos [1]	Cos [2]
−10				
9	$+.01 \cdot 10^{-6}$			
8	.02			
7	.02			
6	.04	$-.01 \cdot 10^{-6}$		
5	.06	.02	$-.01 \cdot 10^{-6}$	
4	.07	.03	.02	
3	.08	.02	.04	
2	+.07	−.01	.07	
− 1	.00	.00	.05	$-.01 \cdot 10^{-6}$
0	$-.04 \cdot 10^{-6}$	$-.04 \cdot 10^{-6}$	$-.04 \cdot 10^{-6}$	$.04 \cdot 10^{-6}$
+ 1	+.02	.00	+.01	−.32
2	.08	−1.09	.02	+1.09.
3	.10	.54	.02	.27
4	.08	.36	.01	.11
5	.07	.25	.01	.06
6	.04	.17	$+.01 \cdot 10^{-6}$	.03
7	.03	.10		.02
8	.02	.05		$+.01 \cdot 10^{-6}$
9	.01	.03		
+10	$+.01 \cdot 10^{-6}$	$-.01 \cdot 10$		

1) Coefficients of the sines and cosines $\overline{M}_2^{(i)} + \frac{1}{2} \varkappa_2^{(i)} (\overline{y}_3' + \underline{y}_3') - M_j$.

2) Coefficients of the sines and cosines $\overline{M}_6^{(i)} + \frac{1}{2} \varkappa_6^{(i)} (\overline{y}_3' + \underline{y}_3') - M_j$

C. Table of Secular Terms

The secular term in $k^2 m_j \int\limits_{t=t_0}^{t} \frac{\partial}{\partial \lambda} (\widetilde{\omega}_j)\, dt$ is equal to $+''.1233324.\, t\partial$;

The secular term in $km_j \int\limits_{t=t_0}^{t} \frac{\partial}{\partial \mu_1} (\widetilde{\omega}_j)\, dt$ is equal to $-.0059341 \cdot 10^{-6}.\, t\partial$;

The secular term in $km_j \int\limits_{t=t_0}^{t} \frac{\partial}{\partial \mu_2} (\widetilde{\omega}_j)\, dt$ is equal to $-.0023618 \cdot 10^{-6}.\, t\partial$.

118

Table 8.

Preliminary Comparison of Interpolational-analytical Theory of Motion of Ceres With Observations.

Epoch Эпоха	$\Delta\alpha$	$\Delta\delta$	Epoch Эпоха	$\Delta\alpha$	$\Delta\delta$
1 - 1801 янв. 21.5	+2720″	+1667″	5 - 1867 май 26.5	−1717″	+442″
10 - 1804 окт. 1.5	+2045	+ 779	4 - 1922 апр. 20.5	−1867	+685
2 - 1811 февр. 24.5	+2465	+ 459	3 - 1931 март 15.0	−1446	+787
4 - 1862 апр. 21.5	− 380	+ 20	9 - 1937 сент. 16.0	−1003	−354

[Key for months same as in Table 1]

Table 9.

Comparison With Observations of the Interpolational-analytical Theory of the Motion of Ceres With Corrected Values of the Osculating Elements

Epoch Эпоха	$\Delta\alpha$	$\Delta\delta$	Epoch Эпоха	$\Delta\alpha$	$\Delta\delta$
1 - 1801 янв. 21.5	−57″	−178″	5 - 1867 май 26.5	+261″	−203″
10 - 1804 окт. 1.5	+49	− 12	4 - 1922 апр. 20.5	− 2	+ 90
2 - 1811 февр. 24.5	+28	+ 20	3 - 1931 март 15.0	+115	− 46
4 - 1862 апр. 21.5	+12	+108	9 - 1937 сент. 16.0	− 11	+130

Mean square deviations: 98" in α and 115" in δ

[Key for months same as in Table 1]

Table

Comparison With Observations of the Theory of the Motion of Ceres Constructed by V. F. Proskurin.

	Epoch Эпоха	$\Delta\alpha$	$\Delta\delta$		Epoch Эпоха	$\Delta\alpha$	$\Delta\delta$
1	1801 янв. 21.5	+90″	+57″	2	1889 февр. 4.5	−113″	+ 7″
11	1809 ноябрь 3.5	+61	+44	6	1895 июнь 22.5	− 80	+ 20
10	1818 окт. 13.5	+20	+26	12	1897 дек. 19.5	−130	− 7
9	1827 сент. 26.5	−20	+ 2	12	1906 дек. 6.98268	−105	− 25
9	1836 сент. 10.5	−34	− 1	11	1915 ноябрь 15.5	− 95	− 24
8	1845 авг. 29.5	−43	− 2	9	1924 сент. 17.0	− 28	+ 12
5	1853 май 5.5	+85	−42	3	1931 март 15.0	−223	−110
4	1862 апр. 21.5	+87	−34	9	1937 сент. 16.0	− 78	+ 9
3	1871 март 19.5	− 9	+ 8	8	1946 авг. 29.0	−153	+ 18
2	1880 февр. 28.5	−54	+12				

Mean square deviations : 97" in α and 36" in δ

- END -